Lecture 2: Simple OLS Regression Estimation

Introduction to Econometrics,2025 Spring

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Review the previous lecture

Causal Inference and RCT

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- The existence of selection bias makes social science more difficult than science.
- Although RCTs is a powerful tool for economists, every project or topic can **NOT** be carried on by it.
- This is the reason why modern econometrics exists and develops. The main job of econometrics is using **non-experimental** data to making convincing causal inference.

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- Let's start our exciting journey from it.

OLS Estimation: Simple Regression

Question: Class Size and Student's Performance

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$$Testscore = f(ClassSize)$$

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- Unfortunately, the function form is always unknown.

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  - parametric: we have to suppose the basic form of the function, then to find values of some *unknown parameters* to determine the specific function form.
- Both methods need to use **samples** to inference **populations** in our random and unknown world.

Question: Class Size and Student's Performance

- Suppose we choose *parametric* method, then we just need to know the real value of a **parameter** β_1 to describe the relationship between Class Size and Test Scores

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- Next step, we have to suppose specific forms of the function $f(\cdot)$, still two categories: linear and non-linear
- And we start to use the *simplest* function form: a **linear** equation, which is graphically **a straight line**, to summarize the relationship between two variables.

$$Test\ score = \beta_0 + \beta_1 \times Class\ size$$

where β_1 is actually the the slope and β_0 is the intercept of the straight line.

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- It also depends on **other factors** such as
  - Student background
  - Quality of the teachers
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  - Quality of text books
  - Random deviation
- So the equation describing the linear relation between Test score and Class size is **better** written as

$$Test\,score_i = \beta_0 + \beta_1 \times Class\,size_i + u_i$$

where u_i lumps together all **other factors** that affect average test scores.

Terminology for Simple Regression Model

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- Where
  - Y_i is the dependent variable(Test Score)
  - X_i is the independent variable or regressor(Class Size or Student-Teacher Ratio)
  - $\beta_0 + \beta_1 X_i$ is the **population regression line** or the **population regression function**

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- Population regression function is the relationship that holds between Y and X **on average over the population.**

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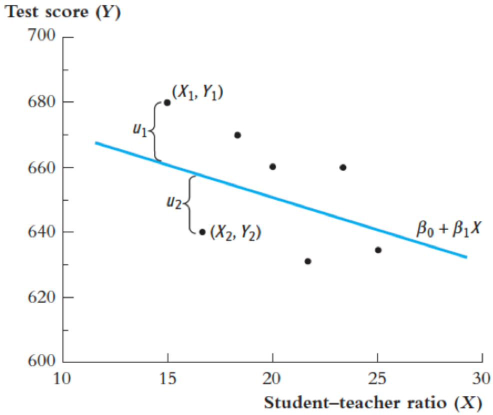
- u_i is the error term which contains all the other factors **besides** X that determine the value of the dependent variable, Y, for a specific observation, i.

Graphics for Simple Regression Model

FIGURE 4.1 Scatterplot of Test Score vs. Student–Teacher Ratio (Hypothetical Data)

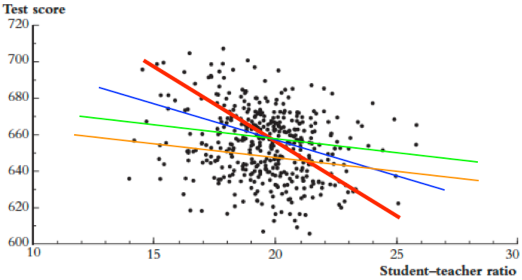
The scatterplot shows hypothetical observations for seven school districts. The population regression line is $\beta_0 + \beta_1 X$. The vertical distance from the i^{th} point to the population regression line is $Y_i - (\beta_0 + \beta_1 X_i)$, which is the population error term u_i for the i^{th} observation.



How to find the "best" fitting line?

- In general we don't know β_0 and β_1 which are parameters of **population regression function** but have to calculate them using a bunch of data: **the sample**.

FIGURE 4.2 Scatterplot of Test Score vs. Student–Teacher Ratio (California School District Data)



Data from 420 California school districts. There is a weak negative relationship between the student–teacher ratio and test scores: The sample correlation is −0.23.

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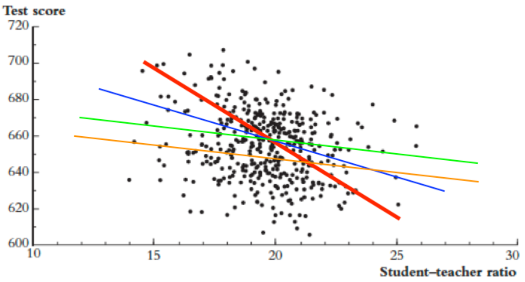

- So how to find the line that fits the data best?

The Ordinary Least Squares Estimator (OLS)

The OLS estimator

- Chooses the **best** regression coefficients so that the estimated regression line is as close as possible to the observed data, where closeness is measured by the sum of the squared mistakes made in predicting Y given X.

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- The predicted value of Y_i given X_i using these estimators is $b_0 + b_1 X_i$, or $\hat{\beta}_0 + \hat{\beta}_1 X_i$ formally denotes as $\hat{Y}_i$, thus

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

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- The estimators of the slope and intercept that *minimize the sum of the squares* of $\hat{u}_i$,thus

$$\underset{b_0,b_1}{arg\,min} \sum_{i=1}^{n} \hat{u}_i^2 = \underset{b_0,b_1}{min} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i)^2$$

are called the ordinary least squares (OLS) estimators of β_0 and β_1.

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- OLS minimizes sum of squared prediction mistakes:

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Step 1: OLS estimator of β_0

- Recall the sample mean of Y_i is

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OLS estimator of β_0:

$$b_0 \equiv \hat{\beta}_0 = \overline{Y} - b_1\overline{X}$$

Step 2: OLS estimator of β_1

$$\frac{\partial}{\partial b_1} \sum_{i=1}^{n} \hat{u}_i^2 = -2 \sum_{i=1}^{n} X_i (Y_i - b_0 - b_1 X_i) = 0$$

Step 2: OLS estimator of β_1

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&\Rightarrow \sum_{i=1}^{n}X_i[(Y_i - \overline{Y}) - b_1(X_i - \overline{X})] = 0 \\
&\Rightarrow \sum_{i=1}^{n}X_i(Y_i - \overline{Y}) - b_1\sum_{i=1}^{n}X_i(X_i - \overline{X}) = 0
\end{aligned}
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- **Some Algebraic Facts**

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&= \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \overline{Y} - n\overline{X}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) + n\overline{XY}
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$$= \sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \overline{Y} - n\overline{X}(\frac{1}{n}\sum_{i=1}^{n} Y_i) + n\overline{XY}$$

$$= \sum_{i=1}^{n} X_i (Y_i - \overline{Y})$$

Step 2: OLS estimator of β_1

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- **By a similar reasoning, we could obtain**

$$\sum_{i=1}^{n}(X_i-\overline{X})(X_i-\overline{X})=\sum_{i=1}^{n}X_i(X_i-\overline{X})=\sum_{i=1}^{n}(X_i-\overline{X})X_i$$

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$$b_1 \equiv \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})(X_i - \overline{X})}$$

Some Algebraic of $\hat{u}_i$

- **Recall the F.O.C**

$$\frac{\partial}{\partial b_0} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i)^2 = 0$$
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- We obtain two intermediate formulas

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- **Obtain the values of OLS estimator for a certain data,**

$$\hat{\beta}_1 = -2.28 \; and \; \hat{\beta}_0 = 698.9$$

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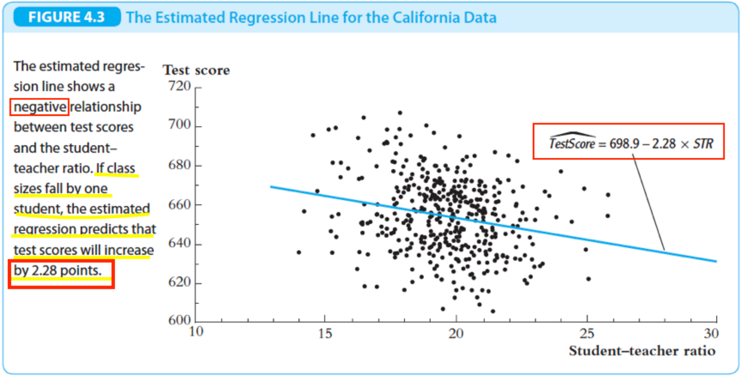

Measures of Fit: The R^2

- Because the variation of Y can be summarized by a statistic: **Variance**,so the total variation of Y_i, which are also called as the **total sum of squares** (TSS), is:

$$TSS = \sum_{i=1}^{n}(Y_i - \overline{Y})^2$$

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- The **explained sum of squares** (ESS): $ESS = \sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2$
- The **sum of squared residuals** (SSR): $\sum_{i=1}^{n}(\hat{Y}_i - Y_i)^2 = \sum_{i=1}^{n} \hat{u}_i^2$

Measures of Fit: The R^2

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&= \sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2 + \sum_{i=1}^{n}\hat{u}_i^2 \\
&= ESS + SSR
\end{aligned}$$

Measures of Fit: The R^2

R^2 or the coefficient of determination

R^2 or the coefficient of determination, is the fraction of the sample variance of Y_i explained/predicted by X_i

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{SSR}{TSS}$$

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- So $0 \leq R^2 \leq 1$, it measures that how much can the variations of Y be explained by the variations of X_i in share.
- Question: If R-squares is bigger, is the regression better?
- Answer: Not necessarily,especially when we make **causal inference** in cross-sectional data.

The Least Squares Assumptions

The Linear Regression Model

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Linear Regression Model

Two random variables Y_i and X_i, their relationship can satisfy the linear regression equation, thus

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- This is not a required assumption. We will extend the model to be nonlinear later on.

Assumption 1: Conditional Mean is Zero

Assumption 1: Zero conditional mean of the errors given X

The error, u_i has expected value of 0 given any value of the independent variable

$$E[u_i \mid X_i = x] = 0$$

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Implications of Assumption 1

With the Iterated Expectation Law, we can obtain an extra implicit assumption about u_i, thus

$$E(u_i) = E(E(u_i|X_i)) = 0$$

- It seems that the assumption is too strong, but given that the linear regression model have a intercept β_0, which means that we could always make the assumption true by redefining the intercept.

Assumption 1: Conditional Mean is Zero

- **An *weaker* condition that u_i and X_i are uncorrelated:**

$$Cov[u_i, X_i] = E[u_i X_i] = 0$$

Covariance and Conditional Mean

Although $Cov[u_i, X_i] = 0 \nRightarrow E[Y_i|X_i]$**, we have**

$$Cov[u_i, X_i] \neq 0 \Rightarrow E[u_i|X_i] \neq 0$$

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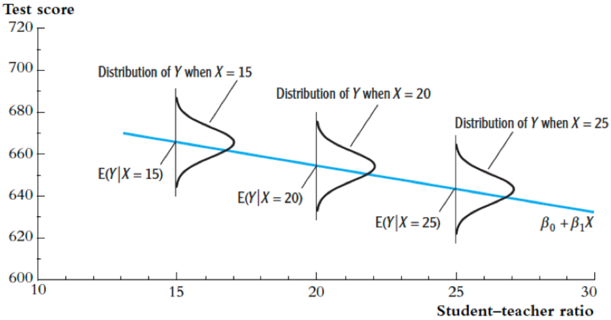
$$Cov[u_i, X_i] \neq 0 \Rightarrow E[u_i|X_i] \neq 0$$

- if u_i and X_i are **correlated**, then Assumption 1 is violated.
- Equivalently, the **population regression line** is the conditional mean of Y_i given X_i , thus

$$E[Y_i|X_i] = \beta_0 + \beta_1 X_i$$

Assumption 1: Conditional Mean is Zero

FIGURE 4.4 The Conditional Probability Distributions and the Population Regression Line



The figure shows the conditional probability of test scores for districts with class sizes of 15, 20, and 25 students. The mean of the conditional distribution of test scores, given the student–teacher ratio, $E(Y|X)$, is the population regression line. At a given value of X, Y is distributed around the regression line and the error, $u = Y - (\beta_0 + \beta_1 X)$, has a conditional mean of zero for all values of X.

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We have a i.i.d random sample of size , $\{(X_i, Y_i), i = 1, ..., n\}$ from the population regression model above.

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- And it generally won't hold in other data structures.
  - time-series, cluster samples and spatial data.

Assumption 3: Large outliers are unlikely

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It states that observations with values of X_i, Y_i or both that are far outside the usual range of the data(Outlier) are unlikely. Mathematically, it assume that X and Y have nonzero finite fourth moments.

- Large outliers can make OLS regression results misleading.

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- One source of large outliers is data entry errors, such as a typographical error or incorrectly using different units for different observations.

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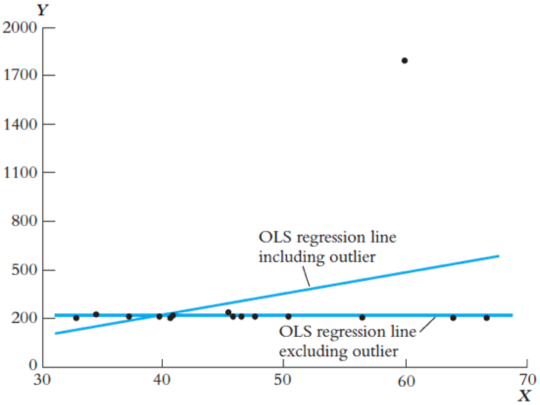
- Large outliers can make OLS regression results misleading.
- One source of large outliers is data entry errors, such as a typographical error or incorrectly using different units for different observations.
- Data entry errors aside, the assumption of finite kurtosis is a plausible one in many applications with economic data.

Assumption 3: Large outliers are unlikely

FIGURE 4.5 The Sensitivity of OLS to Large Outliers

This hypothetical data set has one outlier. The OLS regression line estimated with the outlier shows a strong positive relationship between *X* and *Y*, but the OLS regression line estimated without the outlier shows no relationship.



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- OLS is an **estimator**: it is a machine that we plug data into and we get out estimates.
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- Let's discuss these characteristics of OLS in the next section.

Properties of the OLS Estimators

The OLS estimators

- **Question of interest: What is the effect of a change in X_i(Class Size) on Y_i(Test Score)**

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

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Properties of the OLS estimator: unbiasedness

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- **Then we have: if β_1 is unbiased, then β_0 is also unbiased.**

Properties of the OLS estimator: unbiasedness

- **Remind we have**

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

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Properties of the OLS estimator: unbiasedness

- **Continued**

$$E[\hat{\beta}_1] = E\left[\frac{\sum(X_i - \bar{X})(\beta_0 + \beta_1 X_i + u_i - (\beta_0 + \beta_1 \overline{X} + \overline{u}))}{\sum(X_i - \bar{X})(X_i - \bar{X})}\right]$$

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$$= E\left[\frac{\beta_1 \sum(X_i - \bar{X})(X_i - \overline{X})}{\sum(X_i - \bar{X})(X_i - \bar{X})}\right] + E\left[\frac{(X_i - \overline{X})(u_i - \overline{u})}{\sum(X_i - \bar{X})(X_i - \bar{X})}\right]$$

Properties of the OLS estimator: unbiasedness

- **Continued**

$$
\begin{aligned}
E[\hat{\beta}_1] &= E\Bigg[\frac{\sum(X_i - \bar{X})(\beta_0 + \beta_1 X_i + u_i - (\beta_0 + \beta_1 \overline{X} + \overline{u}))}{\sum(X_i - \bar{X})(X_i - \bar{X})}\Bigg] \\
&= E\Bigg[\frac{\sum(X_i - \bar{X})(\beta_1(X_i - \overline{X}) + (u_i - \overline{u}))}{\sum(X_i - \bar{X})(X_i - \bar{X})}\Bigg] \\
&= E\Bigg[\frac{\beta_1 \sum(X_i - \bar{X})(X_i - \overline{X})}{\sum(X_i - \bar{X})(X_i - \bar{X})}\Bigg] + E\Bigg[\frac{(X_i - \overline{X})(u_i - \overline{u})}{\sum(X_i - \bar{X})(X_i - \bar{X})}\Bigg] \\
&= \beta_1 + E\Bigg[\frac{\sum(X_i - \bar{X})(u_i - \overline{u})}{\sum(X_i - \bar{X})(X_i - \bar{X})}\Bigg]
\end{aligned}
$$

Review: Conditional Expectation Function(CEF)

- Expectation(for a continuous r.v.)

$$E(y) = \int yf(y)dy$$

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$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

- Conditional Expectation Function: the Expectation of Y conditional on X is

$$E(y|x) = \int y f_{Y|X}(y|x) dy$$

Review: Properties of CEF

Let X, Y, Z are random variables; $a, b \in \mathbb{R}$; $g(\cdot)$ is a real valued function, then we have

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- $E[a \mid Y] = a$
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- **If X and Y are independent, then $E[Y \mid X] = E[Y]$**
- $E[Yg(X) \mid X] = g(X)E[Y \mid X]$. **In particular,** $E[g(Y) \mid Y] = g(Y)$

Review: the Law of Iterated Expectations(LIE)

the Law of Iterated Expectations

It states that an unconditional expectation can be written as the unconditional average of conditional expectation function.

$$E(Y_i) = E[E(Y_i|X_i)]$$

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$$E(Y_i) = E[E(Y_i|X_i)]$$

and it can easily extend to

$$E(g(X_i)Y_i) = E[E(g(X_i)Y_i|X_i)] = E[g(X_i)E(Y_i|X_i)]$$

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$$E[E(Y|X)] = \int E(Y|X = u) f_X(u) du$$

Proof: the Law of Iterated Expectation(LIE)

- **Prove it by a continuous variable way**

Proof

$$
\begin{aligned}
E[E(Y|X)] &= \int E(Y|X=u) f_X(u)du \\
&= \int \left[\int t f_Y(t|X=u)dt \right] f_X(u)du
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&= \int t \Big[\int f_{XY}(u,t)du \Big] dt
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Conditional Expectation and Covariance

- **Please prove if** $E(Y|X) = 0 \Rightarrow Cov(X, Y) = 0$

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- **Because $\sum(X_i - \bar{X})(u_i - \overline{u}) = \sum(X_i - \bar{X})u_i$, so**

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&= \beta_1 + \sum E\left[g(X_i)E[u_i|X_i]\right] \because \text{LIE}
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- **Then we can obtain**

$$E\big[\hat{\beta}_1\big] = \beta_1 \; if \; E[u_i|X_i] = 0$$

- **Both** β_0 **and** β_1 **are unbiased on the condition of Assumption 1.**

Properties of the OLS estimator: Consistency

- **Notation:** $\hat{\beta}_1 \xrightarrow{p} \beta_1$ **or** $plim\hat{\beta}_1 = \beta_1$**, so**

$$plim\hat{\beta}_1 = plim\left[\frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})(X_i - \bar{X})}\right]$$

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- Then we could obtain

$$plim \hat{\beta}_1 = plim \left[\frac{\frac{1}{n-1} \sum (X_i - \bar{X})(Y_i - \bar{Y})}{\frac{1}{n-1} \sum (X_i - \bar{X})(X_i - \bar{X})} \right] = plim \left(\frac{s_{xy}}{s_x^2} \right)$$

where s_{xy} and s_x^2 are sample covariance and sample variance.

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where s_{xy} and s_x^2 are sample covariance and sample variance.

Recall: Sample Variance and Sample Covariance

$$s_x^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$$

$$s_{xy} = \frac{1}{n-1}\sum(X_i - \bar{X})(Y_i - \bar{Y})$$

Math Review: Continuous Mapping Theorem

- **Continuous Mapping Theorem**: For every continuous function $g(t)$ and random variable X:

$$plim(g(X)) = g(plim(X))$$

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- **Example**:

$$plim(X + Y) = plim(X) + plim(Y)$$

$$plim(\frac{X}{Y}) = \frac{plim(X)}{plim(Y)} \; if \; plim(Y) \neq 0$$

Properties of the OLS estimator: Consistency

- **Base on L.L.N(the law of large numbers) and random sample(i.i.d)**

$$s_X^2 \xrightarrow{p} \sigma_X^2 = Var(X)$$

$$s_{xy} \xrightarrow{p} \sigma_{XY} = Cov(X, Y)$$

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- **Base on L.L.N(the law of large numbers) and random sample(i.i.d)**

$$s_X^2 \xrightarrow{p} \sigma_X^2 = Var(X)$$

$$s_{xy} \xrightarrow{p} \sigma_{XY} = Cov(X, Y)$$

- **Combining with Continuous Mapping Theorem,then we obtain the OLS estimator $\hat{\beta}_1$,when $n \longrightarrow \infty$**

$$plim\hat{\beta}_1 = plim\left(\frac{s_{xy}}{s_x^2}\right) = \frac{Cov(X_i, Y_i)}{Var(X_i)}$$

Properties of the OLS estimator: Consistency

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&= \beta_1 + \frac{Cov(X_i, u_i)}{Var(X_i)}
\end{aligned}
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- **Then we could obtain**

$$plim\hat{\beta}_1 = \beta_1 \; if \; E[u_i|X_i] = 0$$

Wrap Up: Unbiasedness vs Consistency

- **Unbiasedness** & **Consistency** both rely on $E[u_i|X_i] = 0$

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- **Unbiasedness** & **Consistency** both rely on $E[u_i|X_i] = 0$
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Wrap Up: Unbiasedness vs Consistency

- **Unbiasedness** & **Consistency** both rely on $E[u_i|X_i] = 0$
- **Unbiasedness** implies that $E[\hat{\beta}_1] = \beta_1$ for a certain sample size n.("small sample")
- **Consistency** implies that the distribution of $\hat{\beta}_1$ becomes more and more **tightly distributed** around β_1 if the sample size n becomes larger and larger.("large sample" ")

Wrap Up: Unbiasedness vs Consistency

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- **Unbiasedness** implies that $E[\hat{\beta}_1] = \beta_1$ for a certain sample size n.("small sample")
- **Consistency** implies that the distribution of $\hat{\beta}_1$ becomes more and more **tightly distributed** around β_1 if the sample size n becomes larger and larger.("large sample" ")
- Additionally,you could prove that $\hat{\beta}_0$ is likewise **Unbiased** and **Consistent** on the condition of **Assumption 1**.

Sampling Distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$: Recalll of $\overline{Y}$

- Firstly, Let's recall: Sampling Distribution of $\overline{Y}$

Sampling Distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$: Recalll of $\overline{Y}$

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- Because $Y_1, ..., Y_n$ are i.i.d. and μ_Y is the mean of the population,then for L.L.N,we have

$$E(\overline{Y}) = \mu_Y$$

Sampling Distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$: Recalll of $\overline{Y}$

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$$\overline{Y} \sim N(\mu_Y, \frac{\sigma_Y^2}{n})$$

- Therefore, the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ could have similar sample distributions *when three least squares assumptions hold.*

Sampling Distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$: Expectation

- **Likewise as $\bar{Y}$,the sample distribution of β_1 or β_0 in a large sample can also** *approximates to a normal distribution* **based on the Central Limit theorem(C.L.T)**

$$\hat{\beta}_1 \sim N(\beta_1, \sigma^2_{\hat{\beta}_1})$$
$$\hat{\beta}_0 \sim N(\beta_0, \sigma^2_{\hat{\beta}_0})$$

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- The **expectation** of the OLS estimators is by the **unbiasedness** of the OLS estimators.It implies that

$$E[\hat{\beta}_1] = \beta_1 \; and \; E[\hat{\beta}_0] = \beta_0$$

Sampling Distribution of $\hat{\beta}_1$ and $\hat{\beta}_0$: Variance

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- The **variance** of the OLS estimators can be shown as follows:

$$\sigma^2_{\hat{\beta}_1} = \frac{1}{n}\frac{Var[(X_i - \mu_x)u_i]}{[Var(X_i)]^2})$$
$$\sigma^2_{\hat{\beta}_0} = \frac{1}{n}\frac{Var(H_i u_i)}{(E[H_i^2])^2})$$

Where $H_i = 1 - [\frac{\mu_x}{E[X_i]}]X_i$

Sampling Distribution of $\hat{\beta}_1$ and $\hat{\beta}_0$: Variance

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Where $H_i = 1 - [\frac{\mu_x}{E[X_i]}]X_i$

- *Please prove it by yourself(Hint: refer to Appendix)*

Sampling Distribution $\hat{\beta}_1$ in large-sample

- **We have shown that**

$$\sigma^2_{\hat{\beta}_1} = \frac{1}{n} \frac{Var[(X_i - \mu_x)u_i]}{[Var(X_i)]^2})$$

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- An intuition:The **variation** of X_i is very important.

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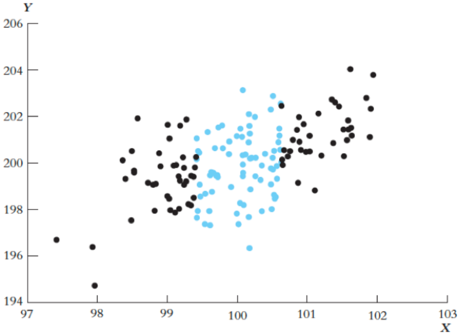
$$\sigma^2_{\hat{\beta}_1} = \frac{1}{n}\frac{Var[(X_i - \mu_x)u_i]}{[Var(X_i)]^2})$$

- An intuition:The **variation** of X_i is very important.
  - Because if $Var(X_i)$ is *small*, it is difficult to obtain an accurate estimate of the effect of X on Y which implies that $Var(\hat{\beta}_1)$ is *large*.

Variation of X

FIGURE 4.6 The Variance of $\hat{\beta}_1$ and the Variance of X

The colored dots represent a set of X_i's with a small variance. The black dots represent a set of X_i's with a large variance. The regression line can be estimated more accurately with the black dots than with the colored dots.



- When more **variation** in X_i, then there is more information in the data that you can use to fit the regression line.

In a Summary

Under 3 least squares assumptions, the OLS estimators will be

- **unbiased**

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- **unbiased**
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- *more variation in X, more accurate estimation*

Simple OLS and RCT

OLS Regression and RCT

- We learned RCT is the **“golden standard”** for causal inference.Because it can naturally eliminate **selection bias**.

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OLS Regression and RCT

- We learned RCT is the **“golden standard”** for causal inference.Because it can naturally eliminate **selection bias**.
- So far, we did not discuss the relationship between RCT and OLS regression, which means that we can not be sure that the result from an OLS regression can be explained as “causal”.
- Instead of using a continuous regressor X, the regression where D_i is a binary variable, a so-called **dummy variable**, will help us to unveil the relationship between RCT and OLS regression.

Regression when X is a Binary Variable

- For example, we may define D_i as follows:

$$D_i = \begin{cases} 1 \text{ if } STR \text{ in } i^{th} \text{ school district} < 20 \\ 0 \text{ if } STR \text{ in } i^{th} \text{ school district} \geq 20 \end{cases} \tag{4.2}$$

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- The regression can be written as

$$Y_i = \beta_0 + \beta_1 D_i + u_i \quad (4.1)$$

Regression when X is a Binary Variable

- More precisely, the regression model now is

$$TestScore_i = \beta_0 + \beta_1 D_i + u_i \tag{4.3}$$

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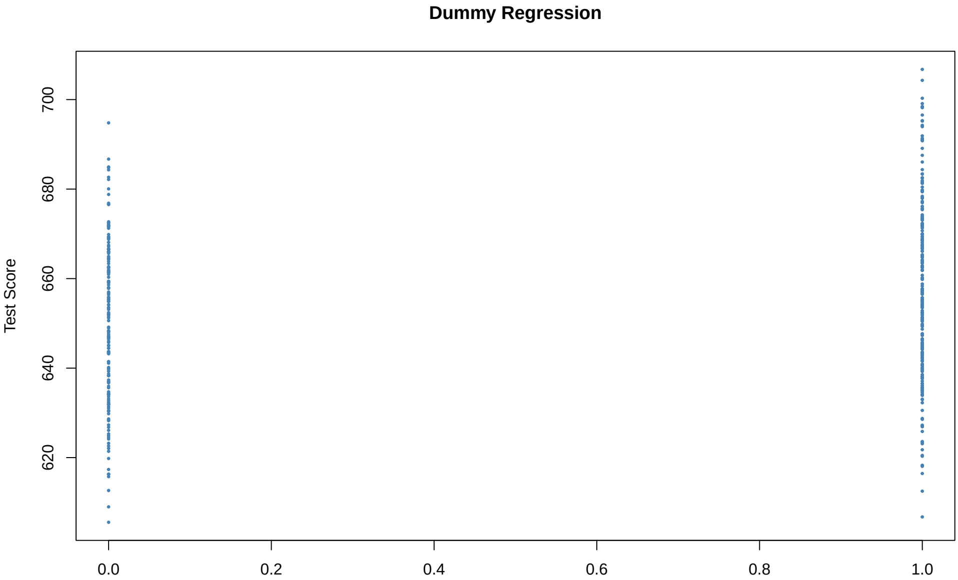
  - With D as the regressor, it is not useful to think of β_1 as a slope parameter.
  - Since $D_i \in \{0, 1\}$, i.e., we only observe two discrete values instead of a continuum of regressor values.
- There is no continuous line depicting the conditional expectation function $E(TestScore_i | D_i)$ since this function is solely defined for x-positions 0 and 1.

Class Size and STR



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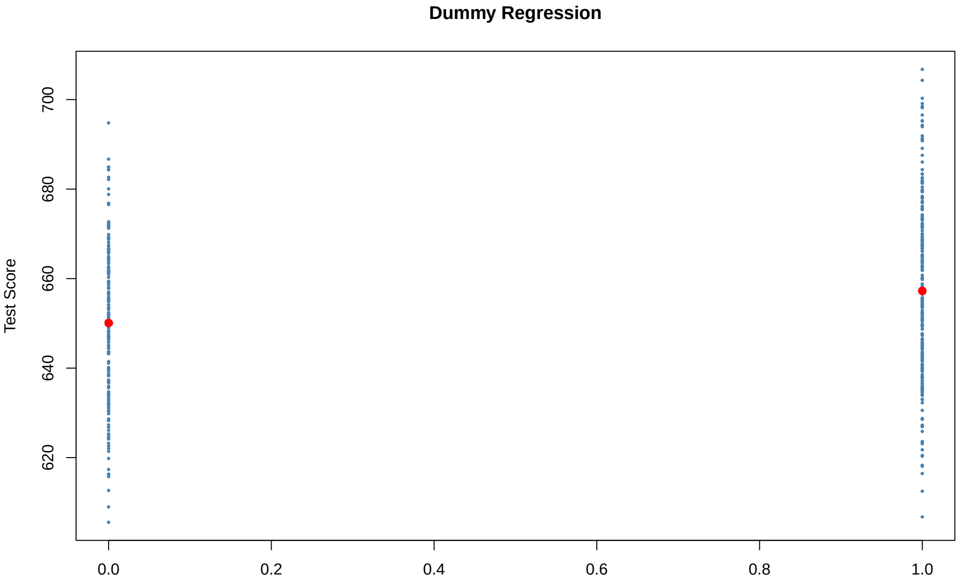

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  - $E(Y_i|D_i = 1) = \beta_0 + \beta_1$ **where** STR **is above** 20

Regression when X is a Binary Variable

- Therefore, the interpretation of the coefficients in this regression model is as follows:
  - $E(Y_i|D_i = 0) = \beta_0$, so β_0 is the expected test score in districts where $D_i = 0$ where STR is below 20.
  - $E(Y_i|D_i = 1) = \beta_0 + \beta_1$ where STR is above 20
- Thus, β_1 is **the difference in group specific expectations**, i.e., the difference in expected test score between districts with $STR < 20$ and those with $STR \geq 20$,

$$\beta_1 = E(Y_i|D_i = 1) - E(Y_i|D_i = 0)$$

.

Causality and OLS

- **Let us recall, the individual treatment effect**

$$ICE = Y_{1i} - Y_{0i} = \delta_i \quad \forall i$$

Causality and OLS

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$$\rho = E(\delta_i) \ or \ \rho = E(\delta_i | D = 1)$$

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- Either way, the treatment effect is a constant, i.e., it does not depend on the individual.
- Our OLS regression function is to estimate a constant treatment effect ρ, thus

$$\mathbf{Y}_i = \underbrace{\alpha}_{E[\mathbf{Y}_{0i}]} + \mathbf{D}_i \underbrace{\rho}_{\mathbf{Y}_{1i} - \mathbf{Y}_{0i}} + \underbrace{\eta_i}_{\mathbf{Y}_{0i} - E[\mathbf{Y}_{0i}]}$$

Causality and OLS

- Now write out the conditional expectation of Y_i for both levels of D_i

$$E\left[\mathbf{Y}_i \mid \mathbf{D}_i = 1\right] = E\left[\alpha + \rho + \eta_i \mid \mathbf{D}_i = 1\right] = \alpha + \rho + E\left[\eta_i | \mathbf{D}_i = 1\right]$$

$$E\left[\mathbf{Y}_i \mid \mathbf{D}_i = 0\right] = E\left[\alpha + \eta_i \mid \mathbf{D}_i = 0\right] = \alpha + E\left[\eta_i \mid \mathbf{D}_i = 0\right]$$

Causality and OLS

- Now write out the conditional expectation of $\mathbf{Y}_i$ for both levels of $\mathbf{D}_i$

$$E[\mathbf{Y}_i \mid \mathbf{D}_i = 1] = E[\alpha + \rho + \eta_i \mid \mathbf{D}_i = 1] = \alpha + \rho + E[\eta_i | \mathbf{D}_i = 1]$$

$$E[\mathbf{Y}_i \mid \mathbf{D}_i = 0] = E[\alpha + \eta_i \mid \mathbf{D}_i = 0] = \alpha + E[\eta_i \mid \mathbf{D}_i = 0]$$

- Take the difference

$$E[\mathbf{Y}_i \mid \mathbf{D}_i = 1] - E[\mathbf{Y}_i \mid \mathbf{D}_i = 0] = \rho + \underbrace{E[\eta_i | \mathbf{D}_i = 1] - E[\eta_i \mid \mathbf{D}_i = 0]}_{\text{Selection bias}}$$

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Causality and OLS

- Again, our estimate of the **treatment effect** (ρ) is only going to be as good as our ability to shut down the **selection bias**.
- *Selection bias in regression model:* $E[\eta_i|\mathbf{D}_i = 1] - E[\eta_i \mid \mathbf{D}_i = 0]$
- There is something in our disturbance η_i that is affecting Y_i and is also correlated with D_i.

Simple OLS Regression v.s. RCT

- In a simple regression model, OLS estimators are just a generalizing continuous version of RCT when least squares assumptions are hold.

Simple OLS Regression v.s. RCT

- In a simple regression model, OLS estimators are just a generalizing continuous version of RCT when least squares assumptions are hold.
- Ideally,regression is a way to control observable confounding factors, which assume the source of selection bias is only from the difference in observed characteristics.

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- But in contrast to RCT, in observational studies, researchers cannot control the assignment of treatment into a treatment group versus a control group,which means that the two groups are **incomparable**.

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- To make two groups comparable, we need to keep treatment and control group "**other thing equal**"in observed characteristics and unobserved characteristics.
- OLS regression is valid only when least squares assumptions are hold.
- In most cases,it is not easy to obtain. We have to know how to make a convincing causal inference when these assumptions are not hold.

Appendix

Sampling Distribution of $\hat{\beta}_1$

- $\hat{\beta}_1$ in terms of regression and errors in following equation

$$\hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(X_i - \overline{X})}$$

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Sampling Distribution of $\hat{\beta}_1$:the numerator

- **The numerator:** $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(u_i - \overline{u})$

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$$\sum_{i=1}^{n}(X_i - \overline{X})(u_i - \overline{u}) = \sum_{i=1}^{n}(X_i - \overline{X})u_i$$

$$\Longrightarrow \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(u_i - \overline{u}) \xrightarrow{p} \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_x)u_i$$

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- **Let** $v_i = (X_i - \mu_x)u_i$

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  - **Based on Assumption 2,** $\sigma_v^2 = Var[(X_i - \mu_x)u_i]$

Sampling Distribution of $\hat{\beta}_1$:the numerator

- **Let** $v_i = (X_i - \mu_x)u_i$
  - **Based on Assumption 1, then** $E(v_i) = 0$
  - **Based on Assumption 2,** $\sigma_v^2 = Var[(X_i - \mu_x)u_i]$
- **Then**

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_x)u_i = \frac{1}{n}\sum_{i=1}^{n} v_i = \bar{v}$$

Sampling Distribution of $\hat{\beta}_1$:the numerator

- **Recall: $\bar{Y}$ to Y_i and based on C.L.T,**

$$\frac{\overline{Y} - 0}{\sigma_{\bar{Y}}} \xrightarrow{d} N(0.1) \; or \; \overline{Y} \xrightarrow{d} N(0, \frac{\sigma_Y^2}{n})$$

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$$\frac{\overline{Y} - 0}{\sigma_{\bar{Y}}} \xrightarrow{d} N(0.1) \; or \; \overline{Y} \xrightarrow{d} N(0, \frac{\sigma_Y^2}{n})$$

- **The** $\bar{v}$ **is the** ***sample mean*** **of** v_i**,based on C.L.T,**

$$\frac{\bar{v} - 0}{\sigma_{\bar{v}}} \xrightarrow{d} N(0.1) \; or \; \bar{v} \xrightarrow{d} N(0, \frac{\sigma_v^2}{n})$$

Sampling Distribution of $\hat{\beta}_1$:the denominator

- **Recall the sample variance of X_i is $s^2_{X_i}$**

$$s^2_{X_i} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

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- Then the denominator,is a variation of **sample variance** of X (except dividing by n rather than $n-1$, which is *inconsequential if* n is large)

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$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(X_i - \overline{X})$$

- Based on discussion of the *sample variance* is a **consistent** estimator of the *population variance*,thus

$$s^2_{X_i} \xrightarrow{p} Var[X_i] = \sigma^2_{X_i}$$

Sampling Distribution of $\hat{\beta}_1$

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$$\hat{\beta}_1 = \beta_1 + \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(u_i - \overline{u})}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(X_i - \overline{X})}$$

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$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(X_i - \overline{X}) \xrightarrow{p} Var[X_i] = \sigma_{X_i}^2$$

- Combining these two results, we have that, *in large samples*

$$\hat{\beta}_1 - \beta_1 \xrightarrow{p} \frac{\bar{v}}{Var[X_i]}$$

Slutsky's Theorem

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Slutsky's Theorem

Suppose that $a_n \xrightarrow{p} a$,where a is a constant,and $S_n \xrightarrow{d} S$.Then

$$a_n + S_n \xrightarrow{d} a + S$$

$$a_n S_n \xrightarrow{d} aS$$

$$\frac{S_n}{a_n} \xrightarrow{d} \frac{S}{a} \; if \; a \neq 0$$

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- Then the sampling distribution of $\hat{\beta}_1$ is

$$\hat{\beta}_1 \xrightarrow{d} N(\beta_1, \sigma^2_{\hat{\beta}_1})$$

where

$$\sigma^2_{\hat{\beta}_1} = \frac{\sigma_v^2}{n[Var(X_i)]^2} = \frac{Var[(X_i - \mu_x)u_i]}{n[Var(X_i)]^2}$$