Lecture 3: Multiple OLS Regression

Introduction to Econometrics,Spring 2025

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Review of the Last Lecture

Simple OLS Formula

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  - X_i is the independent variable or regressor(Class Size or Student-Teacher Ratio)
  - u_i is the error term which **contains all the other factors besides** X that determine the value of the dependent variable, Y, for a specific observation, i.

The OLS Estimator

- The estimators of the slope and intercept that minimize the sum of the squares of $\hat{u}_i$,thus

$$arg\,\underset{b_0,b_1}{min}\sum_{i=1}^{n}\hat{u}_i^2 = \underset{b_0,b_1}{min}\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2$$

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OLS estimator of β_1:

$$b_1 = \hat{\beta}_1 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n} (X_i - \overline{X})(X_i - \overline{X})}$$

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- In such cases, OLS estimators are **biased** and **inconsistent**. Therefore the **causal effect** of X on Y cannot be identified by simple OLS regression.
- To address the selection bias problem, we have to extend the simple OLS regression model in more general settings.

Make Comparison Make Sense

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$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ S \arrow[rr] & & M \end{tikzcd}$$

- **Confounder**, Z, some other factors, affect on smoking and mortality simultaneously.

Case: Smoke and Mortality(Cochran 1968)

Table 1: Death rates(死亡率) per 1,000 person-years

Smoking group	Canada	U.K.	U.S.
Non-smokers(不吸烟)	20.2	11.3	13.5
Cigarettes(香烟)	20.5	14.1	13.5
Cigars/pipes(雪茄/烟斗)	35.5	20.7	17.4

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- It seems that taking cigars is more hazardous than others to the health.

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Table 2: Non-smokers and smokers differ in age

Smoking group	Canada	U.K.	U.S.
Non-smokers(不吸烟)	54.9	49.1	57.0
Cigarettes(香烟)	50.5	49.8	53.2
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- Perhaps the higher observed death rates among cigar smokers are because they're older on average.

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- It naturally relates to the concept of **Conditional Expectation Function**.

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4. Compute the **weighted averages** of the age groups mortality rates for each smoking group using the probability weights.

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	Death rates	Number of	
	Pipe-smokers	Pipe-smokers	Non-smokers
Age 20-50	0.15	11	29
Age 50-70	0.35	13	9
Age +70	0.5	16	2
Total		40	40

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- **Conclusion**: It seems that taking cigarettes is most hazardous, and taking pipes is not different from non-smoking.

Formalization: Covariates

Definition: Covariates

Variable W is predetermined with respect to the treatment D if for each individual i, $W_{0i} = W_{1i}$, i.e., the value of X_i does not depend on the value of D_i. Such characteristics are called *covariates*.

- Covariates are often time invariant (e.g., sex, race), but time invariance is not a necessary condition.

Identification under Independence

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and therefore:

$$E[Y|D=1] - E[Y|D=0] = \underbrace{E[Y_{1i}|D=1] - E[Y_{0i}|D=0]}_{\text{by the switching equation}}$$

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  - Even if $k = 6$, then $3^6 = 729$.Assume that we have 1000 observations, then the average number of observations in each cell is less than 2.

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  - Even if $k = 6$, then $3^6 = 729$.Assume that we have 1000 observations, then the average number of observations in each cell is less than 2.
- Sub-classification is not a feasible method to balance covariates in high-dimensional space.

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- *Selection on Unobservables*
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- The most fundamental tool among them is **multiple regression**, which compares treatment and control subjects who have the same **observable** characteristics in a generalized manner.

Multiple OLS Regression: Introduction

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  - It states that u_i are unrelated to X_i in the sense that,given a value of X_i,the mean of these other factors equals **zero**.

Violation of the 1st Least Squares Assumption

- Recall simple OLS regression equation

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- **Question**: What does u_i represent?
  - Answer: contains **all other factors(variables)** which potentially affect Y_i.
- **Assumption 1**

$$E(u_i|X_i) = 0$$

  - It states that u_i are unrelated to X_i in the sense that,given a value of X_i,the mean of these other factors equals **zero**.
  - But what if u_i is **correlated** with X_i?

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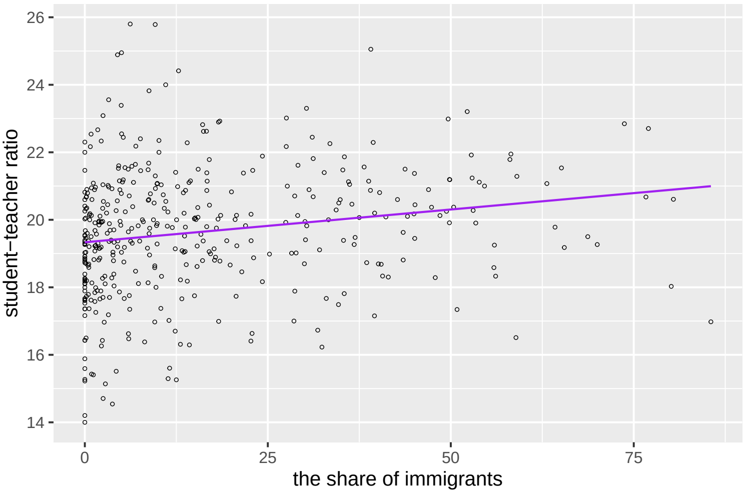

Scatter Plot: The share of immigrants and STR

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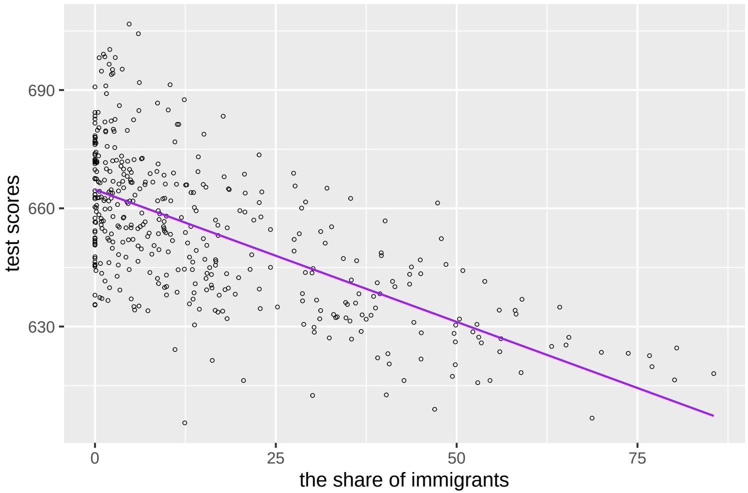

Scatter Plot: The share of immigrants and STR

- higher share of immigrants, lower testscore

The share of immigrants as an Omitted Variable

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- It implies that percentage of English learners is contained in u_i, in turn that **Assumption 1 is violated**.
  - More precisely,the estimates of $\hat{\beta}_1$ and $\hat{\beta}_0$ are **biased** and **inconsistent**.

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    where $v_i = \gamma W_i + u_i$

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- Using the Law of Iterated Expectation(LIE) again, we will obtain the following expression(**Skip these steps which are very *similar* to those for proving unbiasedness of $\hat{\beta}_1$, please prove it by yourself**).

$$E[\hat{\beta}_1] = \beta_1 + \gamma E\left[\frac{\sum(X_i - \bar{X})(W_i - \bar{W})}{\sum(X_i - \bar{X})(X_i - \bar{X})}\right]$$

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- Only if both two conditions above are violated simultaneously, then $\hat{\beta}_1$ is **biased**, which is normally called Omitted Variable Bias(OVB).

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- Recall: simple OLS is consistency when n is large, thus

$$plim\hat{\beta}_1 = \frac{Cov(X_i, Y_i)}{Var(X_i)}$$

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  4. Family income
  5. Percentage of English learners(the share of immigrants)

Omitted Variable Bias: Examples in R

- Regress *Testscore* on *Class size*

```
#>
#> Call:
#> lm(formula = testscr ~ str, data = ca)
#>
#> Residuals:
#>     Min      1Q  Median      3Q     Max
#> -47.727 -14.251   0.483  12.822  48.540
#>
#> Coefficients:
#>             Estimate Std. Error t value Pr(>|t|)
#> (Intercept) 698.9330     9.4675  73.825  < 2e-16 ***
#> str          -2.2798     0.4798  -4.751 2.78e-06 ***
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> Residual standard error: 18.58 on 418 degrees of freedom
#> Multiple R-squared:  0.05124,    Adjusted R-squared:  0.04897
#> F-statistic: 22.58 on 1 and 418 DF,  p-value: 2.783e-06
```

Omitted Variable Bias: Examples in R

- Regress *Testscore* on *Class size* and *the percentage of English learners*

```
#>
#> Call:
#> lm(formula = testscr ~ str + el_pct, data = ca)
#>
#> Residuals:
#>     Min      1Q  Median      3Q     Max
#> -48.845 -10.240  -0.308   9.815  43.461
#>
#> Coefficients:
#>              Estimate Std. Error t value Pr(>|t|)
#> (Intercept) 686.03225    7.41131  92.566  < 2e-16 ***
#> str          -1.10130    0.38028  -2.896  0.00398 **
#> el_pct       -0.64978    0.03934 -16.516  < 2e-16 ***
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> Residual standard error: 14.46 on 417 degrees of freedom
#> Multiple R-squared:  0.4264, Adjusted R-squared:  0.4237
```

Omitted Variable Bias: Examples in R

	Dependent variable:	
	testscr	
	(1)	(2)
str	-2.280^{***}	-1.101^{***}
	(0.480)	(0.380)
el_pct		-0.650^{***}
		(0.039)
Constant	698.933^{***}	686.032^{***}
	(9.467)	(7.411)
Observations	420	420
R^2	0.051	0.426
Note:	*p<0.1; **p<0.05; ***p<0.01	

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- This strategy can be denoted as **controlling** informally, which introduces the more general regression model: **Multiple OLS Regression**.

Multiple OLS Regression: Estimation

Multiple regression model with k regressors

- The multiple regression model is

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + u_i, i = 1, \ldots, n \tag{4.1}$$

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- $X_1, X_2, ...X_k$ are the **independent variables(includes one is our of interest and some control variables)**
- $\beta_i, j = 1...k$ are slope coefficients on X_i corresponding.

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where

- Y_i **is the dependent variable**
- $X_1, X_2, ...X_k$ **are the independent variables(includes one is our of interest and some control variables)**
- $\beta_i, j = 1...k$ **are slope coefficients on** X_i **corresponding.**
- β_0 **is the estimate** ***intercept*****, the value of Y when all** $X_j = 0, j = 1...k$

Multiple regression model with k regressors

- The multiple regression model is

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + u_i, i = 1, \ldots, n \tag{4.1}$$

  where
  - Y_i is the **dependent variable**
  - $X_1, X_2, \ldots X_k$ are the **independent variables(includes one is our of interest and some control variables)**
  - $\beta_i, j = 1 \ldots k$ are slope coefficients on X_i corresponding.
  - β_0 is the estimate *intercept*, the value of Y when all $X_j = 0, j = 1 \ldots k$
  - u_i is the regression *error term*, still all other factors affect outcomes.

Interpretation of coefficients $\beta_i, j = 1...k$

- β_j is **partial (marginal) effect** of X_j on Y.

$$\beta_j = \frac{\partial Y_i}{\partial X_{j,i}}$$

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- it does mean that we are estimate the effect of X on Y when **"other things equal"**, thus the concept of **ceteris paribus**.

OLS Estimation in Multiple Regressors

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$$argmin \sum_{b_0, b_1, ..., b_k} (Y_i - b_0 - b_1 X_{1,i} - ... - b_k X_{k,i})^2$$

where $b_0 = \hat{\beta}_1, b_1 = \hat{\beta}_2, ..., b_k = \hat{\beta}_k$ are estimators.

OLS Estimation in Multiple Regressors

- Similarly in Simple OLS, based on **F.O.C**,the multiple OLS estimators $\hat{\beta}_0, \hat{\beta}_1, ..., \hat{\beta}_k$ are obtained by solving the following **system of normal equations**

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$$\frac{\partial}{\partial b_0} \sum_{i=1}^{n} \hat{u}_i^2 = \sum \left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1,i} - ... - \hat{\beta}_k X_{k,i} \right) \qquad = 0$$

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$$\frac{\partial}{\partial b_1}\sum_{i=1}^{n}\hat{u}_i^2 = \sum\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1,i} - ... - \hat{\beta}_k X_{k,i}\right)X_{1,i} = 0$$

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$$
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\frac{\partial}{\partial b_1}\sum_{i=1}^{n}\hat{u}_i^2 &= \sum\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1,i} - ... - \hat{\beta}_k X_{k,i}\right)X_{1,i} &= 0 \\
\vdots &= \vdots &= \vdots \\
\frac{\partial}{\partial b_k}\sum_{i=1}^{n}\hat{u}_i^2 &= \sum\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1,i} - ... - \hat{\beta}_k X_{k,i}\right)X_{k,i} &= 0
\end{aligned}
$$

OLS Estimation in Multiple Regressors

- Similar to in Simple OLS, the fitted residuals are

$$\hat{u}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1,i} - \ldots - \hat{\beta}_k X_{k,i}$$

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- Therefore, the normal equations also can be written as

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- While it is convenient to transform equations above using **matrix algebra** to compute these estimators, we can use **partitioned regression** to obtain the formula of estimators without using matrix algebra.

Multiple OLS Regression Estimators: Partitioned Regression

Partitioned Regression: OLS estimators

- Suppose our multiple regression model is

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i} + u_i$$

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    1. Regress X_j on $X_1, X_2, ...X_{j-1}, X_{j+1}, X_k$, thus

$$X_{j,i} = \gamma_0 + \gamma_1 X_{1i} + ... + \gamma_{j-1} X_{j-1,i} + \gamma_{j+1} X_{j+1,i} ... + \gamma_k X_{k,i} + v_{ji}$$

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  2. Obtain the **residuals** from the regression above,denoted as $\hat{v}_{ji} \equiv \tilde{X}_{j,i}$

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  2. Obtain the **residuals** from the regression above,denoted as $\hat{v}_{ji} \equiv \tilde{X}_{j,i}$
  3. Regress Y on $\tilde{X}_{j,i}$ to obtain the OLS estimator of β_j.

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  2. Obtain the **residuals** from the regression above,denoted as $\hat{v}_{ji} \equiv \tilde{X}_{j,i}$
  3. Regress Y on $\tilde{X}_{j,i}$ to obtain the OLS estimator of β_j.
- The last step implies that the OLS estimator of β_j can be expressed as follows

$$\hat{\beta}_j = \frac{\sum_{i=1}^{n} \tilde{X}_{ji} Y_i}{\sum_{i=1}^{n} \tilde{X}_{ji}^2}$$

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  - **the first step**: regress $X_{1,i}$ on other regressors Xs, thus

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Partitioned Regression: OLS estimators

- Suppose we want to obtain an expression for $\hat{\beta}_1$.
  - **the first step**: regress $X_{1,i}$ on other regressors $X s$, thus

$$X_{1,i} = \gamma_0 + \gamma_2 X_{2,i} + ... + \gamma_k X_{k,i} + v_i$$

  - **the second step**: obtain the residuals from the regression above, denoted as $\tilde{X}_{1,i} = \hat{v}_{1i}$, thus

$$X_{1,i} = \hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + ... + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}$$

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$$X_{1,i} = \hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + \ldots + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}$$

  - **the third step**: regress Y_i on $\tilde{X}_{1,i}$ to obtain the OLS estimator of β_1, thus

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n \tilde{X}_{1,i} Y_i}{\sum_{i=1}^n \tilde{X}_{1,i}^2}$$

Proof of Partitioned Regression OLS Estimator(1)

- **Recall**: if u_i are the residuals for the Multiple OLS regression equation,thus we have

$$\hat{u}_i = Y_i - \hat{Y}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_{1,i} + \hat{\beta}_2 X_{2,i} + ... + \hat{\beta}_k X_{k,i})$$

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- Then we have

$$\sum \hat{u}_i = \sum \hat{u}_i X_{ji} = 0, j = 1, 2, ..., k$$

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- Then we have

$$\sum \hat{u}_i = \sum \hat{u}_i X_{ji} = 0, j = 1, 2, ..., k$$

- Likewise,$\tilde{X}_{1i} \equiv v_{1i}$ are the residuals for the partitioned regression equation of X_{1i} on $X_{2i}..., X_{ki}$, then we have

$$\sum \tilde{X}_{1i} = \sum \tilde{X}_{1i} X_{2,i} = ... = \sum \tilde{X}_{1i} X_{k,i} = 0$$

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- **Recall**: if u_i are the residuals for the Multiple OLS regression equation,thus we have

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- Then we have

$$\sum \hat{u}_i = \sum \hat{u}_i X_{ji} = 0, j = 1, 2, ..., k$$

- Likewise,$\tilde{X}_{1i} \equiv v_{1i}$ are the residuals for the partitioned regression equation of X_{1i} on $X_{2i}..., X_{ki}$, then we have

$$\sum \tilde{X}_{1i} = \sum \tilde{X}_{1i} X_{2,i} = ... = \sum \tilde{X}_{1i} X_{k,i} = 0$$

- Additionally, because $\tilde{X}_{1,i} = X_{1,i} - \hat{\gamma}_0 - \hat{\gamma}_2 X_{2,i} - ... - \hat{\gamma}_k X_{k,i}$, then we have

$$\sum \hat{u}_i \tilde{X}_{ji} = 0$$

Proof of Partitioned Regression OLS Estimator(2)

$$\frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} =$$

Proof of Partitioned Regression OLS Estimator(2)

$$\frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} = \frac{\sum \tilde{X}_{1,i}(\hat{\beta}_0 + \hat{\beta}_1 X_{1,i} + \hat{\beta}_2 X_{2,i} + ... + \hat{\beta}_k X_{k,i} + \hat{u}_i)}{\sum \tilde{X}_{1,i}^2}$$

Proof of Partitioned Regression OLS Estimator(2)

$$\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}Y_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} = \frac{\sum \tilde{X}_{1,i}(\hat{\beta}_0 + \hat{\beta}_1 X_{1,i} + \hat{\beta}_2 X_{2,i} + \ldots + \hat{\beta}_k X_{k,i} + \hat{u}_i)}{\sum \tilde{X}_{1,i}^2}$$

$$= \hat{\beta}_0 \frac{\sum_{i=1}^{n}\tilde{X}_{1,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + \hat{\beta}_1 \frac{\sum_{i=1}^{n}\tilde{X}_{1,i}X_{1,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + \ldots$$

Proof of Partitioned Regression OLS Estimator(2)

$$
\begin{aligned}
\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}Y_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} &= \frac{\sum \tilde{X}_{1,i}(\hat{\beta}_0+\hat{\beta}_1 X_{1,i}+\hat{\beta}_2 X_{2,i}+\ldots+\hat{\beta}_k X_{k,i}+\hat{u}_i)}{\sum \tilde{X}_{1,i}^2} \\
&= \hat{\beta}_0 \frac{\sum_{i=1}^{n}\tilde{X}_{1,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + \hat{\beta}_1 \frac{\sum_{i=1}^{n}\tilde{X}_{1,i}X_{1,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + \ldots \\
&+ \hat{\beta}_k \frac{\sum_{i=1}^{n}\tilde{X}_{1,i}X_{k,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2} + \frac{\sum_{i=1}^{n}\tilde{X}_{1,i}\hat{u}_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2}
\end{aligned}
$$

Proof of Partitioned Regression OLS Estimator(2)

$$
\begin{aligned}
\frac{\sum_{i=1}^n \tilde{X}_{1,i} Y_i}{\sum_{i=1}^n \tilde{X}_{1,i}^2} &= \frac{\sum \tilde{X}_{1,i}(\hat{\beta}_0 + \hat{\beta}_1 X_{1,i} + \hat{\beta}_2 X_{2,i} + ... + \hat{\beta}_k X_{k,i} + \hat{u}_i)}{\sum \tilde{X}_{1,i}^2} \\
&= \hat{\beta}_0 \frac{\sum_{i=1}^n \tilde{X}_{1,i}}{\sum_{i=1}^n \tilde{X}_{1,i}^2} + \hat{\beta}_1 \frac{\sum_{i=1}^n \tilde{X}_{1,i} X_{1,i}}{\sum_{i=1}^n \tilde{X}_{1,i}^2} + ... \\
&+ \hat{\beta}_k \frac{\sum_{i=1}^n \tilde{X}_{1,i} X_{k,i}}{\sum_{i=1}^n \tilde{X}_{1,i}^2} + \frac{\sum_{i=1}^n \tilde{X}_{1,i} \hat{u}_i}{\sum_{i=1}^n \tilde{X}_{1,i}^2} \\
&= \hat{\beta}_1 \frac{\sum_{i=1}^n \tilde{X}_{1,i} X_{1,i}}{\sum_{i=1}^n \tilde{X}_{1,i}^2}
\end{aligned}
$$

Proof of Partitioned Regression OLS Estimator(3)

Proof(cont'd)

$$\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i}$$

Proof of Partitioned Regression OLS Estimator(3)

Proof(cont'd)

$$\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i} = \sum_{i=1}^{n} \tilde{X}_{1,i} (\hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + \ldots + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i})$$

Proof of Partitioned Regression OLS Estimator(3)

Proof(cont'd)

$$\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i} = \sum_{i=1}^{n} \tilde{X}_{1,i} (\hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + \ldots + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i})$$

$$= \hat{\gamma}_0 \cdot 0 + \hat{\gamma}_2 \cdot 0 + \ldots + \hat{\gamma}_k \cdot 0 + \sum \tilde{X}_{1,i}^2$$

Proof of Partitioned Regression OLS Estimator(3)

Proof(cont'd)

$$\begin{aligned}
\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i} &= \sum_{i=1}^{n} \tilde{X}_{1,i}(\hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + \dots + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}) \\
&= \hat{\gamma}_0 \cdot 0 + \hat{\gamma}_2 \cdot 0 + \dots + \hat{\gamma}_k \cdot 0 + \sum \tilde{X}_{1,i}^2 \\
&= \sum \tilde{X}_{1,i}^2
\end{aligned}$$

Proof of Partitioned Regression OLS Estimator(3)

Proof(cont'd)

$$\begin{aligned}\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i} &= \sum_{i=1}^{n} \tilde{X}_{1,i}(\hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + \dots + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}) \\ &= \hat{\gamma}_0 \cdot 0 + \hat{\gamma}_2 \cdot 0 + \dots + \hat{\gamma}_k \cdot 0 + \sum \tilde{X}_{1,i}^2 \\ &= \sum \tilde{X}_{1,i}^2\end{aligned}$$

- **Then**

$$\frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} =$$

Proof of Partitioned Regression OLS Estimator(3)

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$$
\begin{aligned}
\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i} &= \sum_{i=1}^{n} \tilde{X}_{1,i}(\hat{\gamma}_0 + \hat{\gamma}_2 X_{2,i} + ... + \hat{\gamma}_k X_{k,i} + \tilde{X}_{1,i}) \\
&= \hat{\gamma}_0 \cdot 0 + \hat{\gamma}_2 \cdot 0 + ... + \hat{\gamma}_k \cdot 0 + \sum \tilde{X}_{1,i}^2 \\
&= \sum \tilde{X}_{1,i}^2
\end{aligned}
$$

- **Then**

$$
\frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} =
$$

Proof of Partitioned Regression OLS Estimator(3)

Proof(cont'd)

$$\begin{aligned}\sum_{i=1}^{n}\tilde{X}_{1,i}X_{1,i} &= \sum_{i=1}^{n}\tilde{X}_{1,i}(\hat{\gamma}_0+\hat{\gamma}_2X_{2,i}+\ldots+\hat{\gamma}_kX_{k,i}+\tilde{X}_{1,i})\\ &= \hat{\gamma}_0\cdot 0+\hat{\gamma}_2\cdot 0+\ldots+\hat{\gamma}_k\cdot 0+\sum\tilde{X}_{1,i}^2\\ &= \sum\tilde{X}_{1,i}^2\end{aligned}$$

- **Then**

$$\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}Y_i}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2}=\hat{\beta}_1\frac{\sum_{i=1}^{n}\tilde{X}_{1,i}X_{1,i}}{\sum_{i=1}^{n}\tilde{X}_{1,i}^2}=\hat{\beta}_1$$

Frisch-Waugh-Lowell Theorem

FWL Theorem

The multiple regression model is

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i} + u_i, i = 1, ..., n$$

Then estimator of $\hat{\beta}_1, ..., \hat{\beta}_k$ can be expressed as following

$$\hat{\beta}_j = \frac{\sum_{i=1}^{n} \tilde{X}_{j,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{j,i}^2} = \frac{\sum_{i=1}^{n} \tilde{X}_{j,i} \tilde{Y}_{j,i}}{\sum_{i=1}^{n} \tilde{X}_{j,i}^2} \; for \; j = 1, 2, .., k$$

where $\tilde{X}_{j,i}$ and $\tilde{Y}_{ji}$ are the fitted OLS residuals of the regression X_{ji} and Y_i on all other $X s$ respectively except X_{ji}.

The Intuition of FWL Theorem

Partialling Out

1. First, we regress X_j against the rest of the regressors (and a constant) and keep $\tilde{X}_j$ which is the "part" of X_j that is **uncorrelated** with the other regressors.

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3. $\hat{\beta}_j$ measures the effect of X_1 after the effects of X_2, ..., X_k have been partialled out or netted out.

- FWL Theorem provides a new and important perspective to understand the multiple OLS estimator.

Test Scores and Student-Teacher Ratios(1)

- Now we put one additional control variables into our OLS regression model

$$Testscore = \beta_0 + \beta_1 STR + \beta_2 elpct + u_i$$

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  - First, we regress `str` on `elpct` and keep the residuals $\widetilde{STR}$, thus

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$$STR = \hat{\gamma}_0 + \hat{\gamma}_1 elpct + \widetilde{STR}$$

  - Second, we regress `testscr` on $\widetilde{STR}$ to get the effect of `STR` after controlling for `elpct`.

Test Scores and Student-Teacher Ratios(2)

- **The residuals of the regression of `str` on `elpct` are**

$$\widetilde{STR} = STR - \widehat{STR} = STR - (\hat{\gamma}_0 + \hat{\gamma}_1 elpct)$$

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```
tilde.str <- residuals(lm(str ~ el_pct, data=ca))
tilde.score <- residuals(lm(testscr ~ str+el_pct, data=ca))
sum(tilde.str)  # also is zero
```

```
#> [1] 1.837419e-14
```

```
sum(tilde.str*ca$el_pct) # also should be zero
```

```
#> [1] -3.423928e-13
```

```
sum(tilde.score*tilde.str) # also should be zero
```

```
#> [1] 6.37268e-13
```

Test Scores and Student-Teacher Ratios(3)

- **Multiple OLS estimator in a partitioned way**

$$\hat{\beta}_j = \frac{\sum_{i=1}^{n} \tilde{X}_{j,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{j,i}^2} \ for \ j = 1, 2, .., k$$

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```
sum(tilde.str*ca$testscr)/sum(tilde.str^2)
```

```
#> [1] -1.101296
```

Test Scores and Student-Teacher Ratios(4)

```r
reg3 <- lm(testscr ~ tilde.str,data = ca)
summary(reg3)
```

```
#>
#> Call:
#> lm(formula = testscr ~ tilde.str, data = ca)
#>
#> Residuals:
#>     Min      1Q  Median      3Q     Max
#> -48.693 -14.124   0.988  13.209  50.872
#>
#> Coefficients:
#>             Estimate Std. Error t value Pr(>|t|)
#> (Intercept) 654.1565     0.9254 706.864   <2e-16 ***
#> tilde.str    -1.1013     0.4986  -2.209   0.0277 *
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Test Scores and Student-Teacher Ratios(5)

```r
reg4 <- lm(testscr ~ str+el_pct,data = ca)
summary(reg4)
```

```
#>
#> Call:
#> lm(formula = testscr ~ str + el_pct, data = ca)
#>
#> Residuals:
#>     Min      1Q  Median      3Q     Max
#> -48.845 -10.240  -0.308   9.815  43.461
#>
#> Coefficients:
#>              Estimate Std. Error t value Pr(>|t|)
#> (Intercept) 686.03225    7.41131  92.566  < 2e-16 ***
#> str          -1.10130    0.38028  -2.896  0.00398 **
#> el_pct       -0.64978    0.03934 -16.516  < 2e-16 ***
#> ---
```

Test Scores and Student-Teacher Ratios(6)

	Dependent variable:	
	testscr	
	(1)	(2)
tilde.str	-1.101^{**}	
	(0.499)	
str		-1.101^{***}
		(0.380)
el_pct		-0.650^{***}
		(0.039)
Constant	654.157^{***}	686.032^{***}
	(0.925)	(7.411)
Observations	420	420
Adjusted R^2	0.009	0.424
Note:	*p<0.1; **p<0.05; ***p<0.01	

Measures of Fit in Multiple Regression

Recall: Measures of Fit: The R^2

- **Decompose Y_i into the fitted value plus the residual $Y_i = \hat{Y}_i + \hat{u}_i$**

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- The **sum of squared residuals** (SSR): $\sum_{i=1}^{n}(\hat{Y}_i - Y_i)^2 = \sum_{i=1}^{n} \hat{u}_i^2$
- And

$$TSS = ESS + SSR$$

- The regression R^2 is the fraction of the sample variance of Y_i explained by (or predicted by) the regressors.

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{SSR}{TSS}$$

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- therefore $R_v^2 \geq R_u^2$, thus R^2 that correspinds the regression with one regressor is **less or equal** than R^2 that corresponds to the regression with two regressors.
- This conclusion can be generalized to the case of $k+1$ regressors.

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- At first we would like to know $\sum_{i=1}^{n} \hat{u}_i \hat{v}_i$

$$\sum_{i=1}^{n} \hat{u}_i \hat{v}_i = \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i}) \hat{v}_i$$

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&= \sum_{i=1} \hat{v}_i \hat{v}_i
\end{aligned}
$$

Measures of Fit in Multiple Regression

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- Therefore $R_v^2 \geq R_u^2$, thus R^2 the regression with one regressor is **less or equal** than R^2 that corresponds to the regression with two regressors.

Measures of Fit: The Adjusted R^2

- **the Adjusted** R^2,is a modified version of the R^2 that does not necessarily increase when a new regressor is added.

$$\overline{R^2} = 1 - \frac{n-1}{n-k-1}\frac{SSR}{TSS} = 1 - \frac{s_{\hat{u}}^2}{s_{Y}^2}$$

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  - $\overline{R^2}$ can be negative.
- **Remind**: *neither* R^2 nor $\overline{R^2}$ is NOT the golden criterion for good or bad OLS estimation.

Example: Test scores and Student Teacher Ratios

```
1 . reg testscr str el_pct
```

Source	SS	df	MS
Model	64864.3011	2	32432.1506
Residual	87245.2925	417	209.221325
Total	152109.594	419	363.030056

Number of obs	=	420
F(2, 417)	=	155.01
Prob > F	=	0.0000
R-squared	=	0.4264
Adj R-squared	=	0.4237
Root MSE	=	14.464

testscr	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
str	-1.101296	.3802783	-2.90	0.004	-1.848797	-.3537945
el_pct	-.6497768	.0393425	-16.52	0.000	-.7271112	-.5724423
_cons	686.0322	7.411312	92.57	0.000	671.4641	700.6004

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  - **Assumption 4**: No perfect multicollinearity.

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- Solutions to the dummy variable trap:
    - Omit one of the groups or the intercept

Categoried Variable as Independent Variables

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- For example, we may define D_i as follows:

$$D_i = \begin{cases} 1 & \text{small-size class if } STR \text{ in } i^{th} \text{ school district} < 18 \\ 2 & \text{middle-size class if } 18 \leq STR \text{ in } i^{th} \text{ school district} < 22 \\ 3 & \text{large-size class if } STR \text{ in } i^{th} \text{ school district} \geq 22 \end{cases} \tag{4.5}$$

A Special Case: Categorical Variable as X

- **Naive Solution: a simple OLS regression model**

$$TestScore_i = \beta_0 + \beta_1 D_i + u_i$$

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$$TestScore_i = \beta_0 + \beta_1 D_i + u_i$$

- **Question**: Can you explain the meanning of estimate coefficient β_1?
- **Answer**: It doese not make sense that the coefficient of β_1 can be explained as continuous variables.

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- The first step: turn a categorical variable(D_i) into multiple dummy variables(D_{1i}, D_{2i}, D_{3i})

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$$D_{1i} = \begin{cases} 1 & \text{small-sized class if } STR \text{ in } i^{th} \text{ school district} < 18 \\ 0 & \text{middle-sized class or large-sized class if not} \end{cases}$$

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- **We put these dummies into a multiple regression**

$$TestScore_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 D_{3i} + u_i \tag{4.6}$$

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- Then as a dummy variable as the independent variable in a simple regression The coefficients $(\beta_1, \beta_2, \beta_3)$ represent the effect of every categorical class on $testscore$ respectively.

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- The default intercept term, β_0,represents the large-sized class.Then, the coefficients (β_1, β_2) represent $testscore$ gaps between small_sized, middle-sized class and large-sized class,respectively.

- **regress** *Testscore* **on** *Class size* **and** *the percentage of English learners*

```
#>
#> Call:
#> lm(formula = testscr ~ str + el_pct, data = ca)
#>
#> Residuals:
#>     Min      1Q  Median      3Q     Max
#> -48.845 -10.240  -0.308   9.815  43.461
#>
#> Coefficients:
#>              Estimate Std. Error t value Pr(>|t|)
#> (Intercept) 686.03225    7.41131  92.566  < 2e-16 ***
#> str          -1.10130    0.38028  -2.896  0.00398 **
#> el_pct       -0.64978    0.03934 -16.516  < 2e-16 ***
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '
```

- **add a new variable nel=1-el_pct into the regression**

```
#>
#> Call:
#> lm(formula = testscr ~ str + nel_pct + el_pct, data = ca)
#>
#> Residuals:
#>     Min      1Q  Median      3Q     Max
#> -48.845 -10.240  -0.308   9.815  43.461
#>
#> Coefficients: (1 not defined because of singularities)
#>              Estimate Std. Error t value Pr(>|t|)
#> (Intercept) 685.38247    7.41556  92.425  < 2e-16 ***
#> str          -1.10130    0.38028  -2.896  0.00398 **
#> nel_pct       0.64978    0.03934  16.516  < 2e-16 ***
#> el_pct             NA         NA      NA       NA
#>
```

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Table 4: Class Size and Test Score

	Dependent variable:	
	testscr	
	(1)	(2)
str	-1.101^{***}	-1.101^{***}
	(0.380)	(0.380)
nel_pct		0.650^{***}
		(0.039)
el_pct	-0.650^{***}	
	(0.039)	
Constant	686.032^{***}	685.382^{***}
	(7.411)	(7.416)
Observations	420	420
Adjusted R^2	0.424	0.424
Note:	$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$	

Properties of OLS Estimators in Multiple Regression

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  - **Asymptotic Normality**: $\hat{\beta}_j \sim N(\beta_j, \sigma^2_{\hat{\beta}})$ for $j = 1, 2, ..., k$ in the large sample.

Properties of OLS estimators: Unbiasedness(1)

- Use partitioned regression formula

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}$$

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$$\hat{\beta}_1 = \frac{\sum \tilde{X}_{1,i}(\beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + u_i)}{\sum \tilde{X}_{1,i}^2}$$

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$$= \beta_0 \frac{\sum_{i=1}^{n} \tilde{X}_{1,i}}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} + \beta_1 \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} X_{1,i}}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} + \ldots$$

$$+ \beta_k \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} X_{k,i}}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} + \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} u_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}$$

Properties of OLS estimators: Unbiasedness(2)

- **Because**

$$\sum_{i=1}^{n} \tilde{X}_{1,i} = \sum_{i=1}^{n} \tilde{X}_{1,i} X_{j,i} = 0 \ , j = 2, 3, ..., k$$

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- **Therefore**

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} u_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2}$$

Properties of OLS estimators: Unbiasedness(3)

- **Recall Assumption 1**: $E[u_i|X_{1i}, X_{2i}...X_{ki}] = 0$ and $\tilde{X}_{1i}$ is a function of $X_{2i}...X_{ki}$

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- Recall **Assumption 1**: $E[u_i|X_{1i}, X_{2i}...X_{ki}] = 0$ and $\tilde{X}_{1i}$ is a function of $X_{2i}...X_{ki}$
- Then take expectations of $\hat{\beta}_1$ and **The Law of Iterated Expectations** again

$$
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E[\hat{\beta}_1] = E\left[\beta_1 + \frac{\sum_{i=1}^n \tilde{X}_{1,i} u_i}{\sum_{i=1}^n \tilde{X}_{1,i}^2}\right] &= \beta_1 + E\left[\frac{\sum_{i=1}^n \tilde{X}_{1,i} u_i}{\sum_{i=1}^n \tilde{X}_{1,i}^2}\right] \\
&= \beta_1 + E\left[\frac{\sum_{i=1}^n \tilde{X}_{1,i} E[u_i|X_{1i}...X_{ki}]}{\sum_{i=1}^n \tilde{X}_{1,i}^2}\right] \\
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- Identical argument works for $\beta_2, ..., \beta_k$, thus

$$E[\hat{\beta}_j] = \beta_j \text{ where } j = 1, 2, ..., k$$

Properties of OLS estimators: Consistency(1)

- **Recall**

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$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} \tilde{X}_{1,i} Y_i}{\sum_{i=1}^{n} \tilde{X}_{1,i}^2} = \frac{\frac{1}{n-2}\sum_{i=1}^{n} \tilde{X}_{1i} Y_i}{\frac{1}{n-2}\sum_{i=1}^{n} \tilde{X}_{1i}^2} = \left(\frac{s_{\tilde{X}_1 Y}}{s_{\tilde{X}_1}^2}\right)$$

where $s_{\tilde{X}_1 Y}$ and $s_{\tilde{X}_1}^2$ are the sample covariance of $\tilde{X}_1$ and Y and the sample variance of $\tilde{X}_1$.

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- Combining with *Continuous Mapping Theorem*,then we obtain the partitioned multiple OLS estimator $\hat{\beta}_1$,when $n \longrightarrow \infty$

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$$plim \hat{\beta}_1 = plim \left(\frac{s_{\tilde{X}_1 Y}}{s^2_{\tilde{X}_1}} \right) =$$

Properties of OLS estimators: Consistency(2)

- Base on L.L.N(the law of large numbers) and random sample(i.i.d)

$$s_{\tilde{X}_1^2} \xrightarrow{p} \sigma_{\tilde{X}_1^2} = Var(\tilde{X}_1)$$
$$s_{\tilde{X}_1 Y} \xrightarrow{p} \sigma_{\tilde{X}_1 Y} = Cov(\tilde{X}_1, Y)$$

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- Identical argument works for $\beta_2, ..., \beta_k$,thus

$$plim\hat{\beta}_j = \beta_j \text{ where } j = 1, 2, ..., k$$

Recall: The Distribution of Simple OLS Estimators

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- Specifically, the sampling distribution of $\hat{\beta}_1$ is

$$\hat{\beta}_1 \xrightarrow{d} N(\beta_1, \sigma^2_{\hat{\beta}_1})$$

where

$$\sigma^2_{\hat{\beta}_1} = \frac{Var[(X_i - \mu_x)u_i]}{n[Var(X_i)]^2}$$

The Distribution of Multiple OLS Estimators

- Similarly as in the simple OLS, the multiple OLS estimators are averages of the randomly sampled data, and if the sample size is sufficiently large, the sampling distribution of those averages becomes normal.

$$\hat{\beta}_j = \beta_j + \frac{\left(\sum_{i=1}^{n} \tilde{X}_{ij} u_i\right)}{\left(\sum_{i=1}^{n} \tilde{X}_{ij}^2\right)}$$

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- Here the expression of $Var\left(\sum_{i=1}^{n} \tilde{X}_{ij}^2 u_i\right)$ is a little bit complicated, Then best way mathematically to handle it is using **matrix algebra**, the expressions for the joint distribution of the OLS estimators are deferred to **Chapter 18(SW textbook)**.

Multiple OLS Regression and Causality

Independent Variable v.s Control Variables

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$$\beta_j = \frac{\partial Y_i}{\partial X_{j,i}}$$

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  which means that we are estimate the effect of X on Y when **"other things equal"**, thus the concept of **ceteris paribus**.
- Therefore,other variables in the right hand of equation, we call them **control variables**, which we would like to explicitly **hold fixed** when studying the effect of X_1 or D on Y.

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OLS Regression, Covariates and RCT

- **More specifically,our multiple regression model turns into**

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- We could transform it into as follows

$$Y_i = \alpha + \rho D_i + C_i' \Gamma + u_i$$

where $\alpha = \beta_0, \rho = \beta_1, \Gamma = (\gamma_2, ..., \gamma_k), C_i = (C_{2i}, ..., C_{ki})$

OLS Regression, Covariates and RCT

- Now write out the conditional expectation of Y_i for both levels of D_i conditional on C

$$\begin{aligned}
E\left[\mathbf{Y}_i \mid \mathbf{D}_i = 1, C\right] &= E\left[\alpha + \rho + C'\Gamma + u_i \mid \mathbf{D}_i = 1, C\right] \\
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$$

- Taking the difference

$$
\begin{aligned}
& E\left[\mathbf{Y}_i \mid \mathbf{D}_i=1, C\right]-E\left[\mathbf{Y}_i \mid \mathbf{D}_i=0, C\right] \\
= \rho + & \underbrace{E\left[u_i \mid \mathbf{D}_i=1, C\right]-E\left[u_i \mid \mathbf{D}_i=0, C\right]}_{\text{Selection bias}}
\end{aligned}
$$

OLS Regression, Covariates and RCT

- Again, our estimate of the **treatment effect** (ρ) is only going to be as good as our ability to eliminate the **selection bias**,thus

$$E\left[u_{1i}|\mathbf{D}_i=1, C\right]-E\left[u_{0i} \mid \mathbf{D}_i=0, C\right] \neq 0$$

Conditional Independence Assumption(CIA)

Balancing or controlling covariates C then we can take the treatment D as randomized, thus

$$\left(Y^1, Y^0\right) \perp\!\!\!\perp D|C$$

OLS Regression, Covariates and RCT

- This is the equivalence of the **CIA** assumption, which is also equivalent to the **1st assumption** of Multiple OLS

$$E\left[u_{1i}|\mathbf{D}_i = 1, C\right] - E\left[u_{0i} \mid \mathbf{D}_i = 0, C\right] = E\left[u_{1i}|C\right] - E\left[u_{0i}|C\right]$$

OLS Regression, Covariates and RCT

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- Thus

$$E\left[\mathbf{Y}_i \mid \mathbf{D}_i = 1, C\right] - E\left[\mathbf{Y}_i \mid \mathbf{D}_i = 0, C\right] = \rho$$

Wrap up

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  - If the coefficients of **control variables** (C) are *biased* or *inconsistent*, it does not affect the causal interpretation of the treatment effect.

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- *We will come back soon to discuss the topic in details*(in lecture 7 or 8) .