Lecture 8: Assessing Regression Studies(II)

Introduction to Econometrics,Spring 2026

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May 07 2026

1 Review of previous lectures

2 Internal Validity: Measurement error

3 Simultaneous Causality

4 Functional form misspecification

5 Missing Data and Sample Selection

6 Sample Selection Models

7 Sources of Inconsistency of OLS Standard Errors

8 Magnitude of β_1

Review of previous lectures

Accessing Regression Studies

- The validity of regression studies
  - **Internal** and **External** validity
  - The population and settings are studies and the **generalizability** of the results.
  - The **internal validity** of a regression study is the top priority in causal inference studies.

Internal Validity in OLS Regression

- Suppose we are interested in the causal effect of X_1 on Y and we estimate the following multiple regression model

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i} + u_i, i = 1, ..., n$$

- Internal validity has three components:

1. The estimators of β_1 should be **unbiased and consistent**. This is the most critical aspect.
2. Hypothesis tests and confidence intervals should have the **desired significance level** (at least 5% significant).
3. The value of β_1 should be **large enough** to be meaningful or economically significant.

Threats to Internal Validity

- Threats to internal validity:
  1. **Omitted Variables**
  2. **Misspecification**
  3. **Measurement Error**
  4. **Simultaneous Causality**
  5. **Missing Data and Sample Selection**
  6. **Heteroskedasticity and/or Correlated error terms**
  7. **Significant coefficients or marginal effects**
- In a narrow sense,
  - **Internal Invalidity = endogeneity in the estimation** which is caused by the above 1-5 threats.
- In a broad sense,
  - **Internal Invalidity = 1-5 threats + 6-7 threats**

OLS Regression Estimators in partitioned regression

- **OLS estimator in Multiple OLS**

$$Y_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + ... + \beta_k X_{i,k} + u_i, i = 1, ..., n$$

- **The OLS estimator of β_j is**

$$\hat{\beta}_j = \frac{\sum_{i=1}^{n} \tilde{X}_{ij}, Y_i}{\sum_{i=1}^{n} (\tilde{X}_{ij})^2}$$

- **The asymptotic OLS estimator of β_j**

$$plim\hat{\beta}_j = \frac{Cov(\tilde{X}_{ij}, Y_i)}{Var(\tilde{X}_{ij})}$$

- **Where $\tilde{X}_{ij}$ is the fitted OLS residual of regressing X_{ij} on other regressors,thus**

$$X_{ij} = \hat{\gamma}_0 + \hat{\gamma}_1 X_{i1} + \hat{\gamma}_2 X_{i2} + ... + \hat{\gamma}_{j-1} X_{i,j-1} + \hat{\gamma}_{j+1} X_{i,j+1} + ... + \hat{\gamma}_k X_{i,k} + \tilde{X}_{ij}$$

the Standard Error of $\hat{\beta}$

The Variance of $\hat{\beta}_j$ under Homoskedasticity

$$Var\left(\hat{\beta}_j\right) = \sigma^2_{\hat{\beta}_j} = \frac{\sigma_u^2}{(n-1)s_j^2(1-R_j^2)}$$

- How does the variance of $\hat{\beta}_j$ change with the following factors?

Factors	symbols	$Var\left(\hat{\beta}_j\right)$
the variance of u_i	$\sigma_u^2 \uparrow$	$\uparrow$
the sample variance of X_j	$s_j^2 \uparrow$	$\downarrow$
the R_j^2	$R_j^2 \uparrow$	$\uparrow$
the sample size	$n \uparrow$	$\downarrow$

Control Variables: W

- We will discuss the potential bias or the precise of the OLS estimators when the control variable is as follows:

W	**Irrelevant** to Y	**Relevant** to Y
Uncorrelated with X	**Irrelevant Variables**	**Non-Omitted Variables**
Correlated with X	**Irrelevant Variables**	**Omitted Variables**
Highly-Correlated with X	**(Worse)Irrelevant Variables**	**Omitted Variables and Multicollinearity**

Control Variables: Guides

- **Irrelevant Variables**: Drop
- **Relevant Variables**
  - **Non-Omitted Variables**: Keep
  - **Omitted Variables**: It depends.
- **High Correlation with X**: Be cautious

Good Control v.s Bad Control

- **DAGs** can help us to identify
- Building Blocks
  - **Chains**
  - **Confounders**
  - **Colliders**
- **Good Control**: Block the backdoor path
  - Confounders
- **Bad Control** : Open the backdoor path
  - Colliders and Chains

Tips for Control Variables

1. Identify the variable's category to decide.
2. Use economic theory as a guide.
3. Use **Directed Acyclic Graphs (DAGs)** to visualize and evaluate variable relationships.
4. Gain insights from papers published in reputable journals.

Internal Validity: Measurement error

Introduction

- When a variable is **measured imprecisely**,then it might make OLS estimator biased.
- This bias persists even in very large samples, so the OLS estimator is inconsistent if there is measurement error.
- for example: recall last year's earnings

Types of Measurement errors

There are different types of measurement error

1. Measurement error in the dependent variable Y
   - Less problematic than measurement error in X
   - Usually not a violation of internal validity
   - But leads to less precise estimates
2. Measurement error in the independent variable X(**errors-in-variables bias**)
   - Classical measurement error
   - Measurement error correlated with X
   - Both types of measurement error in X are a violation of internal validity

Measurement error in the dependent variable Y

- Suppose the true population regression model(Simple OLS) is

$$Y_i = \beta_0 + \beta_1 X_i + u_i \quad with \quad E[u_i|X_i] = 0$$

- Because Y is measured with errors, we can not observe Y_i but observe $\tilde{Y}_i$, which is a noisy measure of Y_i,thus

$$\tilde{Y}_i = Y_i + \omega_i$$

- The noisy part of $\tilde{Y}_i$, ω_i, satisfies

$$E[\omega_i|Y_i] = 0$$

- It means that $Cov(\omega_i, Y_i) = 0$ and $Cov(\omega_i, u_i) = 0$,which is a key hypothesis and is called **classical measurement error**
- For example: measurement error due to someone making random mistakes when imputing data in a database.

Measurement error in the dependent variable Y

- **And we can only estimate**

$$\widetilde{Y}_i = \beta_0 + \beta_1 X_i + e_i$$

  where $e_i = u_i + \omega_i$

- **The OLS estimate** $\hat{\beta}_1$ **will be unbiased and consistent because** $E[e_i|X_i] = 0$
- **Nevertheless,the estimate will be less precise because**

$$Var(e_i) > Var(u_i)$$

- **Measurement error in Y is generally less problematic than measurement error in X**

Measurement error in X: classical measurement error

- **The true model is**

$$Y_i = \beta_0 + \beta_1 X_{1i} + u_i$$

  with $E[u_i|X_i] = 0$

- **Due to the classical measurement error,we only have** X^*_{1i} **thus** $X^*_{1i} = X_{1i} + w_i$**,we have to estimate the model is**

$$Y_i = \beta_0 + \beta_1 X^*_{1i} + e_i$$

- **where** $e_i = -\beta_1 w_i + u_i$

Measurement error in X: classical measurement error

- Similar to OVB bias in simple OLS model

$$plim(\hat{\beta}_1) = \frac{Cov(Y_i, X_{1i}^*)}{Var(X_{1i}^*)}$$

Measurement error in X: classical measurement error

- **Similar to OVB bias in simple OLS model**

$$
\begin{aligned}
plim(\hat{\beta}_1) &= \frac{Cov(Y_i, X^*_{1i})}{Var(X^*_{1i})} \\
&= \frac{Cov\left[\beta_0 + \beta_1 X_{1i} + u_i, (X_{1i} + w_i)\right]}{Var(X_{1i} + w_i)}
\end{aligned}
$$

Measurement error in X: classical measurement error

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&= \beta_1 \left(\frac{Var(X_{1i})}{Var(X_{1i}) + Var(w_i)} \right)
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&= \beta_1 \left(\frac{Var(X_{1i})}{Var(X_{1i}) + Var(w_i)} \right) \\
&= \beta_1 \frac{\sigma^2_{X_{1i}}}{\sigma^2_{X_{1i}} + \sigma^2_w}
\end{aligned}
$$

Measurement error in X: classical measurement error

- **Because**

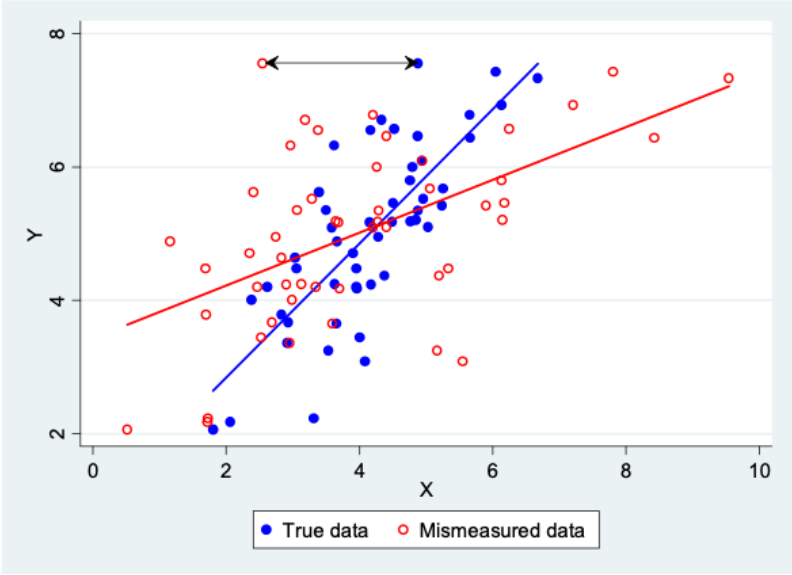
$$0 \leq \frac{\sigma^2_{X_{1i}}}{\sigma^2_{X_{1i}} + \sigma^2_w} \leq 1$$

- **we have**

$$plim(\hat{\beta}_1) = \beta_1 \frac{\sigma^2_{X_{1i}}}{\sigma^2_{X_{1i}} + \sigma^2_w} \leq \beta_1$$

- **The classical measurement error β_1 is biased towards 0, which is also called attenuation bias.**

Measurement error in X: classical measurement error



Solutions to errors-in-variables bias

- The best way to solve the errors-in-variables problem is to get an **accurate measure** of X.
  - Say nothing useful!
- **Instrumental Variables**
  - It relies on having another variable (the "instrumental" variable) that is correlated with the actual value X_i but is uncorrelated with the measurement error. We will discuss it later on.

Simultaneous Causality

Introduction

- So far we assumed that X affects Y, but what if Y also affects X simultaneously ?
  - thus we have $Y_i = \beta_0 + \beta_1 X_1 + u_i$
  - we also have $X_i = \gamma_0 + \gamma_1 Y_1 + v_i$
- Assume that $Cov(v_i, u_i) = 0$, then

$$Cov(X_i, u_i) = Cov(\gamma_0 + \gamma_1 Y_1 + v_i, u_i)$$

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- **Assume that** $Cov(v_i, u_i) = 0$**, then**

$$\begin{aligned} Cov(X_i, u_i) &= Cov(\gamma_0 + \gamma_1 Y_1 + v_i, u_i) \\ &= Cov(\gamma_1 Y_i, u_i) \\ &= Cov(\gamma_1(\beta_0 + \beta_1 X_1 + u_i), u_i) \end{aligned}$$

Introduction

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$$\begin{aligned} Cov(X_i, u_i) &= Cov(\gamma_0 + \gamma_1 Y_1 + v_i, u_i) \\ &= Cov(\gamma_1 Y_i, u_i) \\ &= Cov(\gamma_1(\beta_0 + \beta_1 X_1 + u_i), u_i) \\ &= \gamma_1 \beta_1 Cov(X_i, u_i) + \gamma_1 Var(u_i) \end{aligned}$$

- Simultaneous causality leads to biased & inconsistent OLS estimate.

$$Cov(X_i, u_i) = \frac{\gamma_1}{1 - \gamma_1 \beta_1} Var(u_i)$$

Simultaneous causality bias

- **Substituting $Cov(X_i, u_i)$ in the formula for the $\hat{\beta}_1$**

$$plim\hat{\beta}_1$$

Simultaneous causality bias

- **Substituting $Cov(X_i, u_i)$ in the formula for the $\hat{\beta}_1$**

$$plim\hat{\beta}_1 = \beta_1 + \frac{Cov(X_i, u_i)}{Var(X_{1i})}$$

Simultaneous causality bias

- **Substituting $Cov(X_i, u_i)$ in the formula for the $\hat{\beta}_1$**

$$
\begin{aligned}
plim\hat{\beta}_1 &= \beta_1 + \frac{Cov(X_i, u_i)}{Var(X_{1i})} \\
&= \beta_1 + \frac{\gamma_1 Var(u_i)}{(1 - \gamma_1\beta_1)Var(X_i)} \neq \beta_1
\end{aligned}
$$

- **OLS estimate is inconsistent if simultaneous causality bias exits.**

Solutions to simultaneous causality bias

- The most effective solution is to employ **Instrumental Variables** or other experimental designs.
- **Simultaneous Equations Models** offer a classical alternative, though they are somewhat outdated in modern practice.

Functional form misspecification

Functional form misspecification

- Functional form misspecification also makes the OLS estimator biased and inconsistent.
- It can be seen as an special case of **OVB**,in which the omitted variables are the terms that reflect the missing nonlinear aspects of the regression function.
- It often can be detected by plotting the data and the estimated regression functions, and it can be corrected by using different functional forms.
- More general way is to use **semi-parametric** or **nonparametric** methods.
  - **Matching and Propensity Scores Matching(we will cover it in the next lecture).

Missing Data and Sample Selection

Introduction

- **Missing data** is a common characteristic of economic data sets. It can threaten internal validity if it violates the assumption that our data is a **random sample from the population** of interest.
- In Stata and R, normally values are denoted as "." or "NA" to indicate missing data.
- Whether it poses a threat to internal validity depends on the reason *why the data is missing*.

Three types of missing data

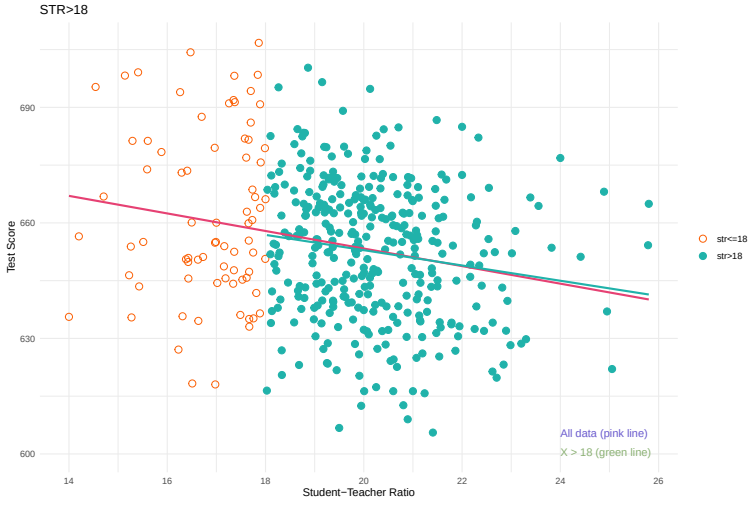
- We consider three types of missing data:
  1. Missing completely at random
     - This does not pose a threat to internal validity. The effect is a reduced sample size, which may impact efficiency but does not introduce bias.
  2. Missing based on X: This shouldn't introduce significant bias into our analysis of the effect of X on Y, as long as the number(or share) of missing data points is relatively small.
     - Ensentially, the key assumption for OLS regression is still hold.

$$E(u_i|X_i) = 0$$

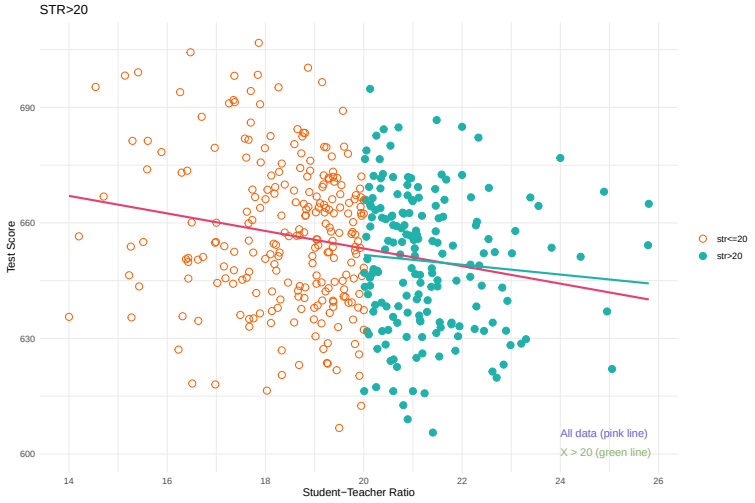
     - And the conditional relationship between Y and X, thus the **causal effect** of X on Y, **remains unbiased** *within the observed data*.

Class Size and Test Score (STR>16)

Class Size and Test Score (STR>18)



Class Size and Test Score(STR>20)



Missing data based on Y

3. Data is missing based on Y: This is the most **problematic type of missing data**. It can introduce **significant bias** into our analysis of the effect of X on Y.

- Essentially, the key assumption for OLS regression is not hold any more in this case.
- Using OLS regression to analyze the effect of X on Y will introduce **bias**.
  - These models are called **Limited dependent variable models**, which normally are non-linear models and can be estimated by **MLE** or **Two-step estimation**.
- Based on the missing data mechanism, we can classify three types of missing data in Y.

Missing data: Censored Data

(A) Data missing(only Y) because of a *selection process* that is related to the value of the dependent variable (Y), which is called censored data(删失数据）.

- An simple example: **the effect of education on income**.
  - Our income data comes from administrative taxation records. However, some households do not report their income, because their income is less than the reporting threshold(like 5000 RMB).
  - Therefore, their income data(Y) is missing, though still have the information of X,like education, age, gender, etc.
- Two **improper** ways to deal with this problem:
  1. **Listwise deletion**: Drop the missing data points.
  2. **Imputation**: Use the top-coded income data (like 5000 RMB) to impute the missing data.
- Both methods will introduce bias into our analysis of the effect of education on income.

Missing data: Censored Data

- A special but useful case: **corner-solution models**.
  - The key feature of the behavior is that the decision can be divided into two parts:
    - The first part is the decision to participate in the behavior.
    - The second part is the decision on the level of the behavior.
- Example: Education on financial investment decision.
  - Many families does not have participation in financial investment. Then the investment data for these families is 0.
  - Other families have participation in financial investment. The investment data can be observed.

Missing data: Truncated Data

(B) Data are total missing(both X and Y) because of a *selection process* that is related to the value of the dependent variable (Y), which is called truncated data.

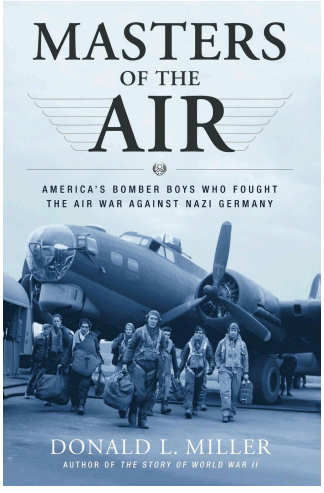
- Example: Innovation Investment on Totol Revenue of firms
  - Our data comes from a survey of firms across years,like **Chinese industrial enterprises database(全部国有及规模以上非国有工业企业数据库)**.
  - Except SOEs, only firms over 500 million RMB in revenue, over 2000 million RMB in revenue after 2011 are included in the sample.
  - Except SOEs, the firms below 500 million RMB in revenue, below 2000 million RMB in revenue after 2011 are totally missing.
- Only use the firms in the sample to estimate the effect of innovation investment on total revenue will introduce **bias**.

Missing data: Sample Selection

(C) Data are missing in Y because of a *selection process* that is related to another variable Z, which is called sample selection data(样本选择数据).

- The selection process depends on another variable Z, meaning that **the selection process can be endogenous**.
- This is the key distinction between **Sample Selection Data** and both **Censored Data** and **Truncated Data**.
- Example: Wage determination of married women(we will cover it in detail later on)
- NOTE: **sample selection or self-selection bias** v.s **selection bias**.

Survivorship Bias from WWII Aircrafts



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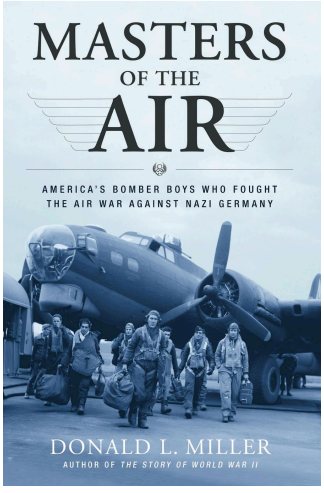
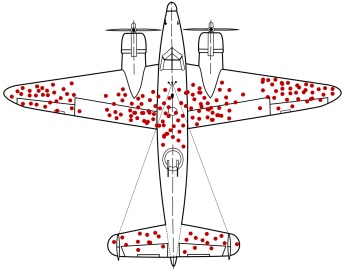


- The bullet holes of a bomber that, crucially, survived

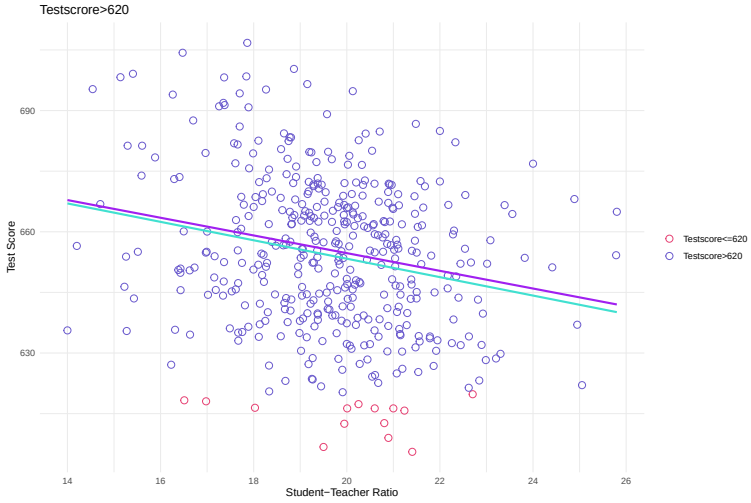


- How to reinforce the armor to increase the survival of allied bombers?
- Which part of the bomber is more important?

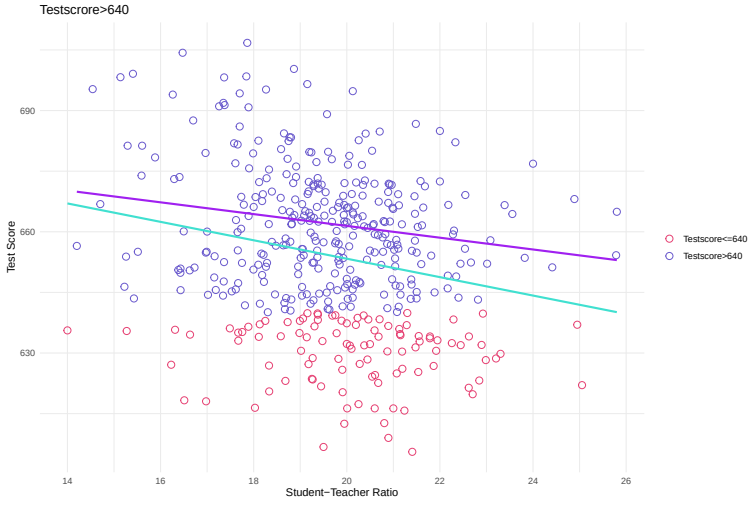
Missing data in Limited Dependent Variable Models

Type	Y availability	X availability	Mechanism
Censored	recorded as $\max(Y^*, c)$	always observed	deterministic threshold on Y^*
Truncated	only if $Y^* > c$	only if $Y^* > c$	observations with $Y^* \leq c$ are dropped
Sample Selection	only if $Z^* > 0$	X (and W) always observed	a *separate* latent equation Z^* governs inclusion

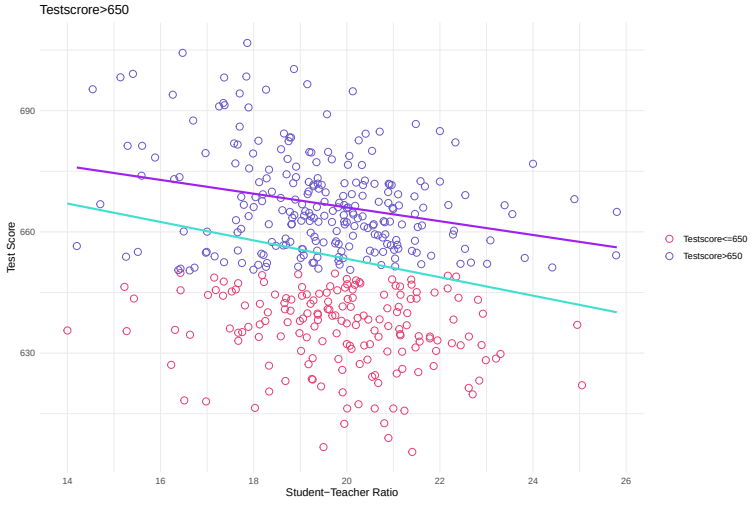
Class Size and Test Score(Test Score>620) — *truncation*



Class Size and Test Score(Test Score>640) — *truncation*



Class Size and Test Score(Test Score>650) — *truncation*



Censored and Truncated Regression Models

- Consider a **latent** variable regression model is $Y_i^* = X_i'\beta + u_i$,here

$$\mathbf{Y}^* = \mathbf{X}'\boldsymbol{\beta} + \mathbf{u} = \begin{bmatrix} Y_1^* \\ Y_2^* \\ \vdots \\ Y_n^* \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & \cdots & X_{k1} \\ 1 & X_{12} & \cdots & X_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1n} & \cdots & X_{kn} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

- Y_i^*: latent dependent variable, which is not observed directly.
- X_i': observed independent variables vector(only one variable we care most)
- β is the parameter vector and u_i is the error term.
- Besides all assumptions of the classical linear regression model, we need an additional assumption to u_i: **Normally distributed**

$$u_i|X_i \sim N(0, \sigma^2)$$

Observation Rules for Latent Y^*

1. Tobit Model: Y_i is observed for everyone, but piles up at some threshold c.

$$Y_i = \max(Y_i^*,\, c) = \begin{cases} Y_i^* & \text{if } Y_i^* > c \\ c & \text{if } Y_i^* \leq c \end{cases}$$

- when $c = 0$, it is the **corner-solution** model.
- when $c > 0$, it is the **censored** model.

2. Truncated regression: observations with $Y_i^* \leq 0$ are *dropped from the sample altogether*; both X_i and Y_i disappear.

- Eitherway, the derivation relies on $E[Y_i \mid X_i, Y_i^* > c]$ — the **conditional-on-selection** expectation function.
- Here **we only cover the case when** $c = 0$,which is the corner-solution model.

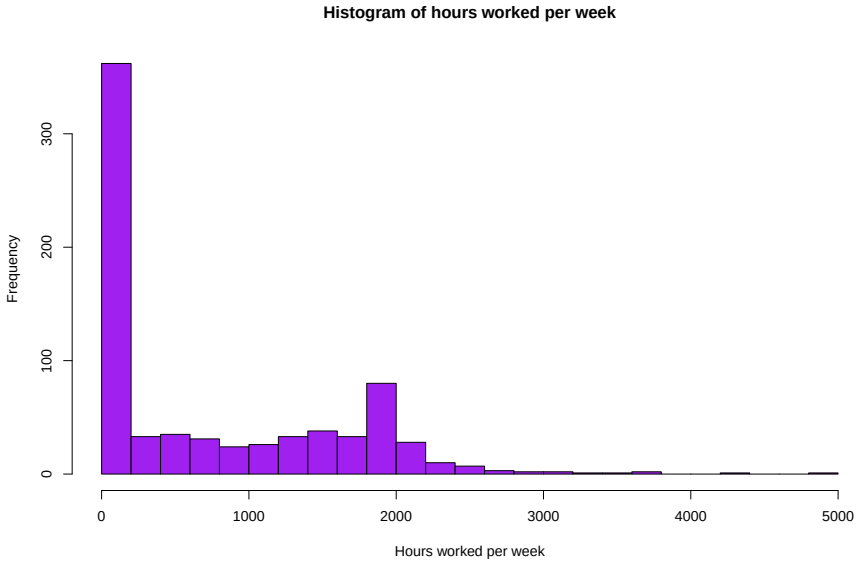
Morz(1987):Labor Supply of Married Women

- Our regression equation is

$$Y_i = \beta_0 + \beta_1 X_i + Z_i' \beta_2 + u_i$$

  - The **dependent variable** Y_i is the **hours worked per week**
  - The **treatment variables** X_i are **education**,
  - The control variables Z_i are **experience, age, and number of children under 6**.
- However, the working hours are not observed for those who do not work, thus we only have a **censored sample**.

Morz(1987):Labor Supply of Married Women



Morz(1987):Labor Supply of Married Women

- We have three options to estimate the effect of education on hours worked by
  1. OLS regression with all observations: include many zero working hours observations.
  2. OLS regression with only hours worked observations: drop many zero working hours observations.
  3. Tobit regression using latent variable model.
- Here Y_i^* can be thought as a measure of intensity of labor supply.
  - Y_i^* is greater, the more hours worked.
  - when $Y_i^* > 0$,we can observe $Y_i = Y_i^*$.
  - when $Y_i^* \leq 0$,we only have $Y_i = 0$.

OLS regression in Censored Sample Data

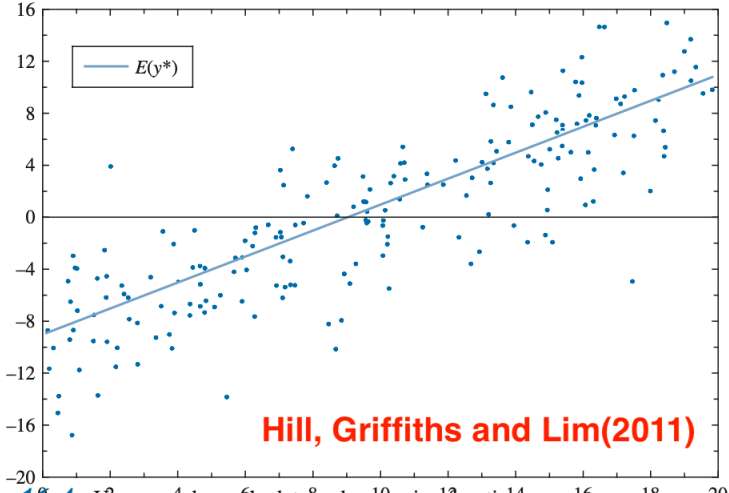


FIGURE 16.4 Uncensored sample data and regression function.

OLS regression in Censored Sample Data

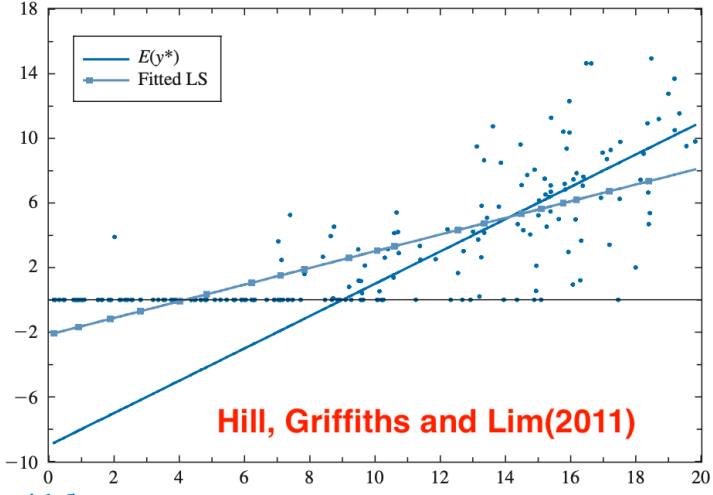


FIGURE 16.5 Censored sample data, and latent regression function and least squares fitted line.

The Expectation of Corner-solution Data

- **Recall the latent variable model:**

$$Y_i^* = X_i'\beta + u_i$$

- **Besides all assumptions of linear regression, error term u_i is normally distributed:**

$$u_i|X_i \sim N(0, \sigma^2)$$

The Expectation of Corner-solution Data

- **Then for the selected sub-population** ($Y_i^* > 0$), **the conditional mean of the** *observed* Y_i **is**

$$\begin{aligned} E[Y_i \mid X_i, Y_i^* > 0] &= E[X_i'\beta + u_i \mid X_i, X_i'\beta + u_i > 0] \\ &= X_i'\beta + E[u_i \mid X_i, u_i > -X_i'\beta] \quad (X_i'\beta \text{ constant given } X_i) \\ &= X_i'\beta + E[u_i \mid u_i > -X_i'\beta] \quad (\text{by } u_i \mid X_i \sim N(0,\sigma^2), \text{ drop } X_i) \\ &= X_i'\beta + \sigma\, E\left[\frac{u_i}{\sigma} \middle| \frac{u_i}{\sigma} > \frac{-X_i'\beta}{\sigma}\right] \end{aligned}$$

- **Note**: the second equality needs the *full distributional* assumption $u_i \mid X_i \sim N(0,\sigma^2)$, *not* merely $E[u_i \mid X_i] = 0$.
- How to get the expectation of $u_i \mid u_i > -X_i'\beta$?

Math Review: Truncated Density Function

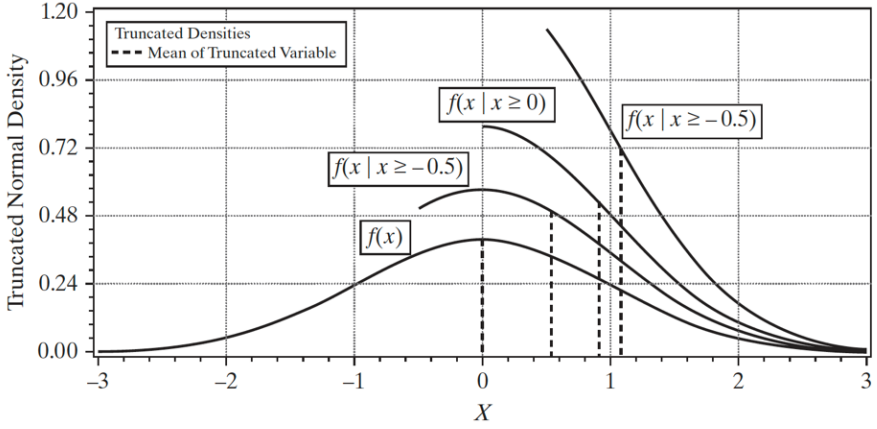
Truncated Density Function

If a continuous random variable X has p.d.f. $f(x)$ and c.d.f. $F(x)$ and a is a constant, then the conditional density function

$$f(x|x > a) = \begin{cases} \frac{f(x)}{1-F(a)} & if\ x > a \\ 0 & if\ x \leq a \end{cases}$$

- Please see the derivation in Appendix Jump to proof.

Math Review: Truncated Density Function



- It amounts merely to scaling the density so that it integrates to **one** over the range above c.

Standard Normal Truncated Density Function

- If X is distributed as standard normal, thus $X \sim N(0,1)$, then the p.d.f and c.d.f are as follow

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

- And c is a scalar, then we can get the *Truncated Density Function* of a R.V. distributed in **Standard Normal**

$$f(x \mid x > c) = \frac{\phi(x)}{1 - \Phi(c)}$$

- The **Expectation** of a *standard-normal truncated R.V.* is

$$E(x \mid x > c) = \frac{\phi(c)}{1 - \Phi(c)} \equiv \lambda(c)$$

(see the proof in Appendix Jump to proof), where $\lambda(c)$ is called the Inverse Mills Ratio（逆米尔斯比率）.

The Expectation of Censored Data at Zero

- Recall we just obtain the expectation of Y_i(observed dependent variable)

$$E[Y_i \mid X_i, Y_i^* > 0] = X_i'\beta + \sigma\, E\left(\frac{u_i}{\sigma} \middle| \frac{u_i}{\sigma} > \frac{-X_i'\beta}{\sigma}\right)$$
$$= X_i'\beta + \sigma\, \lambda\left(\frac{-X_i'\beta}{\sigma}\right)$$

  - Where the $\lambda(\frac{-X_i'\beta}{\sigma})$ is the Inverse Mills Ratio.

- On the **selected sub-sample** ($Y_i^* > 0$), the correctly-specified regression is

$$Y_i = X_i'\beta + \sigma\, \lambda\left(\frac{-X_i'\beta}{\sigma}\right) + \varepsilon_i, \quad i : Y_i^* > 0,$$

i.e. the *truncated regression* specification.

The Bias of OLS on a Censored Sample

- If instead you run the naive model

$$Y_i = X_i'\tilde{\beta} + v_i$$

on the same selected sample,

  - you are absorbing $\sigma\,\lambda(-X_i'\beta/\sigma)$ into the error term v_i, which makes v_i correlated with X_i (because λ is a function of X_i),
  - hence the OLS estimator $\tilde{\beta}$ suffers OVB,
  - and the bias **cannot be removed by adding more controls** — it is intrinsic to the selection.

- **Solution**: Tobit MLE estimates $(\hat{\beta}, \hat{\sigma})$ jointly from both the zeros and the positive observations. we skip technical details here.

Labor supply (Mroz 1987): coefficient estimates

	Dependent variable:		
	hours	hours2	hours
	OLS	*OLS*	*Tobit*
	(1)	(2)	(3)
educ	27.086**	−16.462	73.291***
	(12.240)	(15.581)	(20.475)
exper	48.040***	33.936***	80.535***
	(3.642)	(5.009)	(6.288)
age	−31.308***	−17.108***	−60.768***
	(3.961)	(5.458)	(6.888)
kidsl6	−447.855***	−305.309***	−918.918***
	(58.413)	(96.449)	(111.661)
Constant	1,335.306***	1,829.746***	1,349.876***
	(235.649)	(292.536)	(386.299)
Observations	753	428	753
Adjusted R^2	0.253	0.117	
Log Likelihood			-3,827.143
F Statistic	64.711*** (df = 4; 748)	15.123*** (df = 4; 423)	

Note: $^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Tobit Model in R

- The Tobit coefficients β_j are the marginal effects on the **latent** variable Y^* — they are interpretable, just **not** the marginal effect on the observed Y.
- For policy/economic interpretation, we typically want the marginal effect on the *observed* Y, which requires the **McDonald–Moffitt decomposition**.

Tobit: three marginal effects (McDonald–Moffitt, 1980)

Let $z = X_i'\beta/\sigma$ and $\lambda(\cdot) = \phi(\cdot)/\Phi(\cdot)$. For a continuous regressor X_{ij} Tobit yields **three** distinct marginal effects:

Effect	Formula	Interpretation
Latent	$\dfrac{\partial E[Y^* \mid X]}{\partial X_{ij}} = \beta_j$	Effect on the *desired* (latent) outcome
Conditional on $Y > 0$	$\beta_j \left[1 - \lambda(z)\{z + \lambda(z)\}\right]$	Effect among the *workers* (intensive margin)
Unconditional	$\beta_j \, \Phi(z)$	Population-average effect on observed Y

- **Decomposition**:
  - (i) pushing more women across the participation margin,
  - (ii) raising hours among workers.

The Tobit Model Regression in Graphs

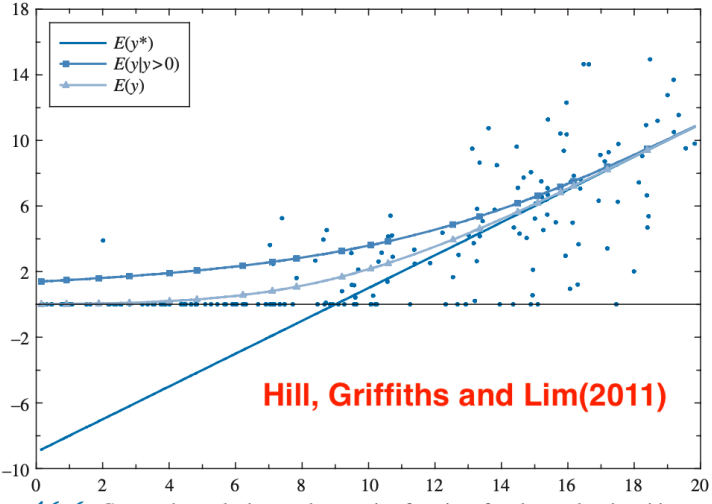


FIGURE 16.6 Censored sample data, and regression functions for observed and positive y-values.

Labor supply (Mroz 1987): AME of education

Estimator	AME on educ	SE
Linear models		
OLS (all obs.)	27.09	(12.24)
OLS (workers only)	-16.46	(15.58)
Tobit (McDonald–Moffitt)		
Tobit: latent (β)	73.29	(21.25)
Tobit: conditional E[Y|Y>0]	31.20	(8.92)
Tobit: unconditional E[Y]	44.37	(12.67)

Note:

AME of `educ`, evaluated at sample means of (exper, age, kidsl6). Tobit AMEs use $\hat{\beta}_{\text{educ}} = 73.29$, $\hat{\sigma} = 1134$, $\hat{z} = \bar{X}'\hat{\beta}/\hat{\sigma} = 0.27 \Rightarrow \Phi(\hat{z}) = 0.61$. SEs for Tobit rows are bootstrap (B=200).

Sample Selection Models

Example: Wage determination of married women

- A Classical Example: wage determination for Married Women

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

  - Y_i is logwage
  - X_i is schooling years
- The sample selection problem arises in that the sample consists only of women who chose to work.
  - If the selection to work is random,then OK.
  - But in reality, married women choose to work probably they are *smarter,more ambitious* and more risk-preferent which normally can not observed or measured in the data(Z).

Heckman Sample Selection Model(I)

- **A two-equation behavioral model**

1. *selection equation*

$$Z_i^* = W_i'\gamma + e_i$$

where Z_i is a latent variable which indicates the propensity of working for a married woman

- **the error term e_i satisfies**

$$E[e_i|W_i] = 0$$

- **Then Z_i is a dummy variable to represent whether a woman to work or not actually,thus**

$$Z_i = \begin{cases} 1 \; if \; Z^* > 0 \\ 0 \; if \; Z^* \leq 0 \end{cases}$$

Heckman Sample Selection Model(II)

2. *outcome equation*

$$Y_i^* = X_i'\beta + u_i$$

- **where the outcome(Y_i^*) can be observed only when Z_i=1 or $Z_i^* > 0$**

$$Y_i^* = \begin{cases} Y_i \; if \; Z_i = 1 \\ 0 \; or \; missing \; if \; Z_i = 0 \end{cases}$$

- **The error term u_i satisfies $E[u_i|X_i] = 0$**

Heckman Sample Selection Model(III)

- The conditional expectation of wages on X_i is

$$E[Y_i^*|X_i] = X_i'\beta$$

- The conditional expectation of wages on X_i is only for women who work($Z^* > 0$)

$$E[Y_i^*|X_i, Z_i^* > 0] = E[Y_i|X_i, Z_i^* > 0]$$

Heckman Sample Selection Model(III)

- **The conditional expectation of wages on X_i is**

$$E[Y_i^*|X_i] = X_i'\beta$$

- **The conditional expectation of wages on X_i is only for women who work($Z^* > 0$)**

$$\begin{aligned} E[Y_i^*|X_i, Z_i^* > 0] &= E[Y_i|X_i, Z_i^* > 0] \\ &= E[X_i'\beta + u_i|X_i, Z_i^* > 0] \end{aligned}$$

Heckman Sample Selection Model(III)

- The conditional expectation of wages on X_i is

$$E[Y_i^*|X_i] = X_i'\beta$$

- The conditional expectation of wages on X_i is only for women who work($Z^* > 0$)

$$\begin{aligned} E[Y_i^*|X_i, Z_i^* > 0] &= E[Y_i|X_i, Z_i^* > 0] \\ &= E[X_i'\beta + u_i|X_i, Z_i^* > 0] \\ &= X_i'\beta + E[u_i|Z_i^* > 0] \end{aligned}$$

Heckman Sample Selection Model(III)

- **The conditional expectation of wages on X_i is**

$$E[Y_i^*|X_i] = X_i'\beta$$

- **The conditional expectation of wages on X_i is only for women who work($Z^* > 0$)**

$$\begin{aligned} E[Y_i^*|X_i, Z_i^* > 0] &= E[Y_i|X_i, Z_i^* > 0] \\ &= E[X_i'\beta + u_i|X_i, Z_i^* > 0] \\ &= X_i'\beta + E[u_i|Z_i^* > 0] \\ &= X_i'\beta + E[u_i|e_i > -W_i'\gamma] \end{aligned}$$

Heckman Sample Selection Model(IV)

- If u_i and e_i is independent, then $E[u_i|e_i > -W_i'\gamma] = 0$, then

$$E[Y_i^*|X_i, Z_i^* > 0] = E[Y_i^*|X_i] = X_i'\beta$$

  - It means using sample-selected data does not make the estimation of β biased.
- But in reality, unobservables in the two equations, thus u_i and e_i, are likely to be **correlated**
  - eg. innate ability,ambitions,
- Instead assume that u_i and e_i are **jointly normal distributed**, which can be standardized easily, thus

$$\begin{pmatrix} u_i \\ e_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_u \\ \mu_e \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{eu} \\ \sigma_{ue} & \sigma_e^2 \end{pmatrix}\right) =$$

Heckman Sample Selection Model(IV)

- If u_i and e_i is independent, then $E[u_i|e_i > -W_i'\gamma] = 0$, then

$$E[Y_i^*|X_i, Z_i^* > 0] = E[Y_i^*|X_i] = X_i'\beta$$

  - It means using sample-selected data does not make the estimation of β biased.
- But in reality, unobservables in the two equations, thus u_i and e_i, are likely to be **correlated**
  - eg. innate ability,ambitions,
- Instead assume that u_i and e_i are **jointly normal distributed**, which can be standardized easily, thus

$$\begin{pmatrix} u_i \\ e_i \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_u \\ \mu_e \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{eu} \\ \sigma_{ue} & \sigma_e^2 \end{pmatrix}\right) = \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \rho\sigma_u \\ \rho\sigma_u & 1 \end{pmatrix}\right)$$

- where we let $\sigma_e^2 = 1$, and ρ is the correlation coefficient between u_i and e_i

Math Review: Two Normal Distributed R.V.s

Two Normal Distributed R.V.s

For any two normal variables (n_0, n_1) with zero mean, we can write $n_1 = \alpha_0 n_0 + \eta$, where $\eta \sim N(0, \sigma_\eta)$ and $E(\eta | n_0) = 0$.Then we have

Math Review: Two Normal Distributed R.V.s

Two Normal Distributed R.V.s

For any two normal variables (n_0, n_1) with zero mean, we can write $n_1 = \alpha_0 n_0 + \eta$, where $\eta \sim N(0, \sigma_\eta)$ and $E(\eta | n_0) = 0$.Then we have

$$\alpha_0 = \frac{Cov(n_0, n_1)}{Var(n_0)}$$

Math Review: Two Normal Distributed R.V.s

Two Normal Distributed R.V.s

For any two normal variables (n_0, n_1) with zero mean, we can write $n_1 = \alpha_0 n_0 + \eta$, where $\eta \sim N(0, \sigma_\eta)$ and $E(\eta | n_0) = 0$.Then we have

$$\alpha_0 = \frac{Cov(n_0, n_1)}{Var(n_0)}$$

or

$$E(n_1 \mid n_0) = \frac{Cov(n_0, n_1)}{Var(n_0)} n_0$$

Math Review: Two Normal Distributed R.V.s

Two Normal Distributed R.V.s

For any two normal variables (n_0, n_1) with zero mean, we can write $n_1 = \alpha_0 n_0 + \eta$, where $\eta \sim N(0, \sigma_\eta)$ and $E(\eta | n_0) = 0$.Then we have

$$\alpha_0 = \frac{Cov(n_0, n_1)}{Var(n_0)}$$

or

$$E(n_1 \mid n_0) = \frac{Cov(n_0, n_1)}{Var(n_0)} n_0$$

Then

$$n_1 = E(n_1 \mid n_0) + \eta = \frac{Cov(n_0, n_1)}{Var(n_0)} n_0 + \eta$$

Heckman Sample Selection Model(V)

- **For two normal variables u_i and e_i with zero means, we have**

$$\alpha_0 = \frac{Cov(u_i, e_i)}{Var(e_i)} = \frac{\sigma_{ue}}{\sigma_e^2}$$

- **Then**

$$u_i = \alpha_0 e_i + \eta = \frac{\sigma_{ue}}{\sigma_e^2} e_i + \eta$$

where $\eta \sim N(0, \sigma_\eta)$ **and** $E(\eta | e_i) = 0$

Heckman Sample Selection Model(VI)

- **Then the conditional expectation of** u_i

$$E[u_i|e_i > -W_i'\gamma]$$

Heckman Sample Selection Model(VI)

- **Then the conditional expectation of** u_i

$$E[u_i|e_i > -W_i'\gamma] = E[\frac{\sigma_{ue}}{\sigma_e^2}e_i + \eta|e_i > -W_i'\gamma]$$

Heckman Sample Selection Model(VI)

- **Then the conditional expectation of** u_i

$$
\begin{aligned}
E[u_i|e_i > -W_i'\gamma] &= E[\frac{\sigma_{ue}}{\sigma_e^2}e_i + \eta|e_i > -W_i'\gamma] \\
&= \frac{\sigma_{ue}}{\sigma_e^2}E[e_i|e_i > -W_i'\gamma] + E[\eta|e_i > -W_i'\gamma]
\end{aligned}
$$

Heckman Sample Selection Model(VI)

- **Then the conditional expectation of u_i**

$$
\begin{aligned}
E[u_i|e_i > -W_i'\gamma] &= E[\frac{\sigma_{ue}}{\sigma_e^2}e_i + \eta|e_i > -W_i'\gamma] \\
&= \frac{\sigma_{ue}}{\sigma_e^2}E[e_i|e_i > -W_i'\gamma] + E[\eta|e_i > -W_i'\gamma] \\
&= \frac{\sigma_{ue}}{\sigma_e^2}E[e_i|e_i > -W_i'\gamma]
\end{aligned}
$$

Heckman Sample Selection Model(VII)

- **Then the conditional expectation of** u_i

$$E[u_i|e_i > -W_i'\gamma] = \frac{\sigma_{ue}}{\sigma_e^2} E[e_i|e_i > -W_i'\gamma]$$

Heckman Sample Selection Model(VII)

- **Then the conditional expectation of** u_i

$$E[u_i|e_i > -W_i'\gamma] = \frac{\sigma_{ue}}{\sigma_e^2}E[e_i|e_i > -W_i'\gamma]$$
$$= \frac{\sigma_{ue}}{\sigma_e}E[\frac{e_i}{\sigma_e}|\frac{e_i}{\sigma_e} > \frac{-W_i'\gamma}{\sigma_e}]$$

Heckman Sample Selection Model(VII)

- **Then the conditional expectation of** u_i

$$
\begin{aligned}
E[u_i|e_i > -W_i'\gamma] &= \frac{\sigma_{ue}}{\sigma_e^2} E[e_i|e_i > -W_i'\gamma] \\
&= \frac{\sigma_{ue}}{\sigma_e} E[\frac{e_i}{\sigma_e}|\frac{e_i}{\sigma_e} > \frac{-W_i'\gamma}{\sigma_e}] \\
&= \frac{\sigma_{ue}}{\sigma_e} \frac{\phi(-W_i'\gamma/\sigma_e)}{1-\Phi(-W_i'\gamma/\sigma_e)}
\end{aligned}
$$

Heckman Sample Selection Model(VII)

- Then the conditional expectation of u_i

$$
\begin{aligned}
E[u_i|e_i > -W_i'\gamma] &= \frac{\sigma_{ue}}{\sigma_e^2} E[e_i|e_i > -W_i'\gamma] \\
&= \frac{\sigma_{ue}}{\sigma_e} E[\frac{e_i}{\sigma_e}|\frac{e_i}{\sigma_e} > \frac{-W_i'\gamma}{\sigma_e}] \\
&= \frac{\sigma_{ue}}{\sigma_e} \frac{\phi(-W_i'\gamma/\sigma_e)}{1-\Phi(-W_i'\gamma/\sigma_e)} \\
&= \frac{\sigma_{ue}}{\sigma_e} \frac{\phi(W_i'\gamma/\sigma_e)}{\Phi(W_i'\gamma/\sigma_e)}
\end{aligned}
$$

for the pdf and cdf of a standard normal dis.

Heckman Sample Selection Model(VII)

- Then the conditional expectation of u_i

$$
\begin{aligned}
E[u_i|e_i > -W_i'\gamma] &= \frac{\sigma_{ue}}{\sigma_e^2} E[e_i|e_i > -W_i'\gamma] \\
&= \frac{\sigma_{ue}}{\sigma_e} E[\frac{e_i}{\sigma_e}|\frac{e_i}{\sigma_e} > \frac{-W_i'\gamma}{\sigma_e}] \\
&= \frac{\sigma_{ue}}{\sigma_e} \frac{\phi(-W_i'\gamma/\sigma_e)}{1-\Phi(-W_i'\gamma/\sigma_e)} \\
&= \frac{\sigma_{ue}}{\sigma_e} \frac{\phi(W_i'\gamma/\sigma_e)}{\Phi(W_i'\gamma/\sigma_e)} \text{ for the pdf and cdf of a standard normal dis.} \\
&= \frac{\sigma_{ue}}{\sigma_e} \lambda(W_i'\gamma/\sigma_e) \text{ where } \lambda(x) = \frac{\phi(x)}{\Phi(x)}
\end{aligned}
$$

Heckman Sample Selection Model(VII)

- Then the conditional expectation of u_i

$$
\begin{aligned}
E[u_i|e_i > -W_i'\gamma] &= \frac{\sigma_{ue}}{\sigma_e^2} E[e_i|e_i > -W_i'\gamma] \\
&= \frac{\sigma_{ue}}{\sigma_e} E[\frac{e_i}{\sigma_e}|\frac{e_i}{\sigma_e} > \frac{-W_i'\gamma}{\sigma_e}] \\
&= \frac{\sigma_{ue}}{\sigma_e} \frac{\phi(-W_i'\gamma/\sigma_e)}{1-\Phi(-W_i'\gamma/\sigma_e)} \\
&= \frac{\sigma_{ue}}{\sigma_e} \frac{\phi(W_i'\gamma/\sigma_e)}{\Phi(W_i'\gamma/\sigma_e)} \text{ for the pdf and cdf of a standard normal dis.} \\
&= \frac{\sigma_{ue}}{\sigma_e} \lambda(W_i'\gamma/\sigma_e) \text{ where } \lambda(x) = \frac{\phi(x)}{\Phi(x)} \\
&= \sigma_\lambda \lambda(W_i'\gamma) \quad for\ \sigma_e = 1\ and\ \sigma_\lambda = \sigma_{ue}
\end{aligned}
$$

Heckman Sample Selection Model(VIII)

- Then the conditional expectation of wages on X_i is only for women who work($Z^* > 0$)

$$E[Y_i^*|X_i, Z_i^* > 0] = E[Y_i|X_i, Z_i = 1] = X_i'\beta + \sigma_\lambda \lambda(W_i'\gamma)$$

  - Turning it into a regression form

$$Y_i = X_i'\beta + \sigma_\lambda \lambda(W_i'\gamma) + u_i$$

- Recall our original wage determination equation

$$Y_i = X_i'\tilde{\beta} + v_i$$

- Likewise, the error term v_i is correlated with the independent variable X_i, thus the OLS estimator $\tilde{\beta}$ will suffer the OVB bias.

Heckman Sample Selection Model(IX)

- It means that if we could include $\lambda(W_i'\gamma)$ as **an additional regressor** into the outcome equation, thus we run

$$Y_i = X_i'\beta + \sigma_\lambda \lambda(W_i'\gamma) + u_i$$

  obtaining the **unbiased** and **consistent** estimate β using a self-selected sample.
- The coefficient before $\lambda(\cdot)$ can be testing significance to indicate whether the term should be included in the regression, in other words, whether the selection should be corrected.

Heckit Model Estimation: a two-step method

1. Estimate selection equation using all observations,thus

$$Z_i = W_i'\gamma + e_I$$

- obtain estimates of parameters $\hat{\gamma}$
- computer the **Inverse Mills Ratio(IMR)** $\dfrac{\phi(W_i'\hat{\gamma})}{\Phi(W_i'\hat{\gamma})} = \hat{\lambda}(W_i'\hat{\gamma})$

2. Estimate the outcome equation using only the selected observations.

$$Y_i = X_i'\beta + \sigma_\lambda \hat{\lambda}(W_i'\hat{\gamma}) + u_i$$

- **Note**: standard error is not right, have to be adjusted because we use $\hat{\lambda}(W_i'\hat{\gamma})$ instead of $\lambda(W_i'\gamma)$ in the estimation.

An Example: Wage Equation for Married Women

TABLE 17.7 Wage Offer Equation for Married Women

Dependent Variable: log(*wage*)		
Independent Variables	**OLS**	**Heckit**
educ	.108 (.014)	.109 (.016)
exper	.042 (.012)	.044 (.016)
$exper^2$	−.00081 (.00039)	−.00086 (.00044)
constant	−.522 (.199)	−.578 (.307)
$\hat{\lambda}$	—	.032 (.134)
Sample size *R*-squared	428 .157	428 .157

Tobit $\subset$ Heckman: when to use which?

	Tobit	Heckit (Heckman)
Selection equation	implicit: same as outcome	explicit: $Z^* = W'\gamma + e$
Coefficients	one set β	two sets γ and β
Exclusion restriction	not needed	**needed in** W for credible identification
Why Y is missing	corner / censoring at 0	endogenous self-selection on a different Z

Tobit $\subset$ Heckman: when to use which?

- Tobit is the special case of Heckman in which the **same** β governs the participation and the intensity decisions: a variable that raises participation must raise hours by the *same* sign and proportion.
- If you suspect that, e.g., young children **discourage working** but, conditional on working, have **little effect on hours**, Tobit is too restrictive — go to Heckman.
- **Empirical practice**: report both; check whether the IMR coefficient σ_λ is significantly different from zero. The exclusion restriction in Heckman is the same idea you will see in **IV** next week.

Wrap Up

- Missing data is a common problem in practice for empirical researchers.
- Missing data can be caused by many reasons, such as **non-response**, **attrition**, **non-sampling error**, etc.
- Missing data can be **missing completely at random (MCAR)**, **missing at random (MAR)**, or **missing not at random (MNAR)**.
  - Normally, missing values in X may not be a serious problem.
  - Missing values in Y are problematic.
- Limited Dependent Variable Models are using to deal with missing data in Y.
  - Tobit model
  - Heckman Sample Selection Model

Sources of Inconsistency of OLS Standard Errors

Introduction

- A different threat to internal validity.Even if the OLS estimator is **consistent** and the sample is large, **inconsistent standard errors** will let you make a **bad judgment** about the effect of the interest in the population.
- There are two main reasons for inconsistent standard errors:

1. Heteroskedasticity: The solution to this problem is to use **heteroskedasticity-robust standard errors** and to construct **F-statistics** using a heteroskedasticity-robust variance estimator.

Sources of Inconsistency of OLS Standard Errors

2. Correlation of the error term across observations:
   - This will not happen if the data are obtained by sampling at random from the population.(**i.i.d**)
   - Sometimes, however, sampling is only **partially random**.
     - When the data are repeated observations on the same entity over time.(**time series**)
     - Another situation in which the error term can be correlated across observations is when sampling is based on a geographical or other group unit.(**cluster**)

- Both situation means that the assumptions

$$Cov(u_i, u_j) \neq 0$$

the second key assumption in OLS is partially violated.

- In this case, the OLS estimator is still **unbiased** and **consistent**, but the standard errors are **inconsistent**.

Clustering Standard Error: A Simple Example

- Suppose we still focus on the topic of class size and student performance, but now the data are collecting on **students** rather than school district.
- Our regression model is

$$TestScore_{ig} = \beta_0 + \beta_1 ClassSize_g + u_{ig}$$

- $TestScore_{ig}$ is the *dependent variable* for student i in class g, with G groups.
- $ClassSize_g$ the *independent variable*(or treatment variable), **varies only at the group level**(class).
- Intuitively,the test score of students in the same class(g) tend to be correlated. Thus

$$Cov[u_{ig}, u_{jg}] = \rho\sigma_u^2$$

where ρ is the intraclass correlation coefficient.

Clustering Standard Error(I)

- **Recall the variance of the OLS estimator:**

$$Var(\hat{\beta}_j) = Var\left(\beta_j + \frac{\sum_{i=1}^{n} \tilde{X}_{ij} u_i}{\sum_{i=1}^{n} \tilde{X}_{ij}^2}\right) = \frac{Var\left(\sum_{i=1}^{n} \tilde{X}_{ij} u_i\right)}{\left(\sum_{i=1}^{n} \tilde{X}_{ij}^2\right)^2}$$

With clustering, we reorganize the sum by clusters:

$$Var(\hat{\beta}_j) = \frac{Var\left(\sum_{g=1}^{G} \sum_{i \in g} \tilde{X}_{ij} u_{ig}\right)}{\left(\sum_{i=1}^{n} \tilde{X}_{ij}^2\right)^2}$$

Clustering Standard Error(II)

- When the sample is clustered, which means that the observations are only randomly sampled across clusters, g and G is the number of clusters.
- Then the numerator of the variance of the OLS estimator is:

$$Var\left(\sum_{i=1}^{n}\tilde{X}_{ij}u_i\right) =$$

Clustering Standard Error(II)

- When the sample is clustered, which means that the observations are only randomly sampled across clusters, g and G is the number of clusters.
- Then the numerator of the variance of the OLS estimator is:

$$Var\left(\sum_{i=1}^{n}\tilde{X}_{ij}u_i\right) = Var\left(\sum_{g=1}^{G}\sum_{i\in g}\tilde{X}_{ij}u_{ig}\right)$$

$$=$$

Clustering Standard Error(II)

- When the sample is clustered, which means that the observations are only randomly sampled across clusters, g and G is the number of clusters.
- Then the numerator of the variance of the OLS estimator is:

$$
\begin{aligned}
Var\left(\sum_{i=1}^{n}\tilde{X}_{ij}u_i\right) &= Var\left(\sum_{g=1}^{G}\sum_{i\in g}\tilde{X}_{ij}u_{ig}\right) \\
&= \sum_{g=1}^{G}Var\left(\sum_{i\in g}\tilde{X}_{ij}u_{ig}\right) \quad \text{for i.i.d sample in g level} \\
&=
\end{aligned}
$$

Clustering Standard Error(II)

- When the sample is clustered, which means that the observations are only randomly sampled across clusters, g and G is the number of clusters.
- Then the numerator of the variance of the OLS estimator is:

$$
\begin{aligned}
Var\left(\sum_{i=1}^{n}\tilde{X}_{ij}u_i\right) &= Var\left(\sum_{g=1}^{G}\sum_{i\in g}\tilde{X}_{ij}u_{ig}\right) \\
&= \sum_{g=1}^{G} Var\left(\sum_{i\in g}\tilde{X}_{ij}u_{ig}\right) \quad \text{for i.i.d sample in g level} \\
&= \sum_{g=1}^{G}\left[\sum_{i\in g}\sum_{k\in g}\tilde{X}_{ij}\tilde{X}_{kj}Cov(u_{ig},u_{kg})\right]
\end{aligned}
$$

Clustering Standard Error(III)

- **Substituting** $Cov(u_{ig}, u_{kg}) = \begin{cases} \sigma_u^2 & \textbf{if } i = k \\ \rho\sigma_u^2 & \textbf{if } i \neq k \end{cases}$:

$$Var(\hat{\beta}_j) = \frac{\sum_{g=1}^{G} \left[\sigma_u^2 \sum_{i \in g} \tilde{X}_{ij}^2 + \rho\sigma_u^2 \sum i \in g \sum_{k \in g, k \neq i} \tilde{X}_{ij} \tilde{X}_{kj} \right]}{\left(\sum_{i=1}^{n} \tilde{X}_{ij}^2 \right)^2}$$

$$= \frac{\sigma_u^2 \sum_{g=1}^{G} \left[\sum_{i \in g} \tilde{X}_{ij}^2 + \rho \sum i \in g \sum_{k \in g, k \neq i} \tilde{X}_{ij} \tilde{X}_{kj} \right]}{\left(\sum_{i=1}^{n} \tilde{X}_{ij}^2 \right)^2}$$

- **This final expression shows how intraclass correlation ρ inflates the variance through the additional cross-product terms.**

Clustering Standard Error(IV)

- **Stata: use option** `vce(cluster clustvar)`**. Where** `clustvar` **is a variable that identifies the groups in which on observables are allowed to correlate.**
- **R: the** `vcovHC()` **function from** `plm` **package**

Magnitude of β_1

Introduction

- The criteria for determining the magnitude of β_1 are as follows:
  - large enough to make sense.
  - **Question**: How large is considered large enough?
- The magnitude of β_1 is not only determined by the actual relationship between X and Y, but also by the units in which X and Y are measured.
- Recall the class size and student performance example, the coefficient β_1 is -2.38, which means that if class size increases by 1, then student performance decreases by 2.38 points.
  - Whether the -2.38 is large or small depends on the scale of the variables and distribution of the data.
- Normally, we compare the magnitude of β_1 to the **mean value of** Y or the **standard deviation of** Y.

Standardized Variables

- **Assume Xs and Y are all continuous variables, then we run a multiple regression model**

$$Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_{i1} + \hat{\beta}_2 X_{i2} + \cdots + \hat{\beta}_k X_{ik} + \hat{u}_i$$

- **Because $\Sigma \hat{u}_i = 0$ and $\overline{Y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}_1 + \cdots + \hat{\beta}_k \bar{X}_k$,then**

$$Y_i - \bar{Y} = \hat{\beta}_1 (X_{i1} - \bar{X}_1) + \hat{\beta}_2 (X_{i2} - \bar{X}_2) + \cdots + \hat{\beta}_k (X_{ik} - \bar{X}_k) + \hat{u}_i$$

- **Then, we obtain following expressions**

$$\frac{Y_i - \bar{Y}}{\sigma_y} = \hat{\beta}_1 \frac{\sigma_{x_1}}{\sigma_y} \frac{(X_{i1} - \bar{X}_1)}{\sigma_{x_1}} + \hat{\beta}_2 \frac{\sigma_{x_1}}{\sigma_y} \frac{(X_{i2} - \bar{X}_2)}{\sigma_{x_2}} + \cdots + \hat{\beta}_k \frac{\sigma_{x_1}}{\sigma_y} \frac{(X_{i2} - \bar{X}_k)}{\sigma_{x_k}} + \frac{\hat{u}_i}{\sigma_y}$$

Standardized Variables

- Then we have a standardized regression model

$$Z_y = \hat{\phi}_1 Z_1 + \hat{\phi}_2 Z_2 + \cdots + \hat{\phi}_k Z_k + v_i$$

  where Z_y denotes the Z-score of Y, Z_1 denotes the **Z-score** of X_1,and so on.

- The estimate coefficients

$$\hat{\phi}_j = (\hat{\sigma}_j / \hat{\sigma}_y)\, \hat{\beta}_j \text{ for } j = 1, \ldots, k$$

- $\hat{\phi}_j$ are traditionally called **standardized coefficients** or **beta coefficients**, which can be explained as if X_j increases by **1 standard deviation**, then Y changes by ϕ **standard deviations**.

Standardized Only One X

- Consider a linear regression model as usual

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + u_i$$

- Or

$$Y_i = \beta_0 + \beta_1 X_{i1} + C\Gamma' + u_i$$

- Where $\Gamma = (\beta_2, ..., \beta_k), C_i = (X_{2i}, ..., X_{ki})$
- If we only standardize X_1 and leave other variables as they are, then the standardized version of X_1 is defined as

$$Z_1 = \frac{X_1 - \bar{X}_1}{\sigma_{x_1}}$$

- Then we have the standardized regression model

Standardized Only One X

- Substitute Z_1 back into the original regression equation in place of X_1, we have

$$\begin{aligned} Y_i =& \beta_0 + \beta_1 \left(\frac{X_1 - \bar{X}_1}{\sigma_{x_1}} \right) + C\Gamma' + u_i \\ =& \beta_0 - \frac{\bar{X}_1}{\sigma_{x_1}} + \beta_1 \frac{X_1}{\sigma_{x_1}} + C\Gamma' + u_i \end{aligned}$$

- Then we have the marginal effect of X_1 on Y as

$$\begin{aligned} \frac{\partial Y}{\partial X_1} &= \beta_1 \frac{1}{\sigma_{x_1}} \\ \Rightarrow \beta_1 &= \frac{\partial Y}{\frac{\partial X_1}{\sigma_{x_1}}} \end{aligned}$$

- The estimate coefficients $\hat{\beta}_1$ is can be interpreted as follows:
  - if X_1 increases by 1 standard deviation, then Y changes by β_1 units.

Standardized Only Y

- If we only standardize Y and leave other variables as they are, then the standardized version of Y is defined as

$$Z_Y = \frac{Y - \bar{Y}}{\sigma_Y}$$

- Then the regression model becomes

$$\begin{aligned} Z_Y &= \beta_0 + \beta_1 X_1 + C\Gamma' + u_i \\ \Rightarrow \frac{Y - \bar{Y}}{\sigma_Y} &= \beta_0 + \beta_1 X_1 + C\Gamma' + u_i \\ \Rightarrow \frac{Y}{\sigma_Y} &= \beta_0 + \frac{\bar{Y}}{\sigma_Y} + \beta_1 X_1 + C\Gamma' + u_i \end{aligned}$$

Standardized Only Y

- **Then we have the marginal effect of X_1 on Y as**

$$\frac{\partial Y}{\partial X_1} = \beta_1 \sigma_y$$

$$\Rightarrow \beta_1 = \frac{\frac{\partial Y}{\sigma_y}}{\partial X_1}$$

- **The estimate coefficients $\hat{\beta}_1$ is can be interpreted as follows:**
  - **if X_1 increases by 1 unit, then Y changes by β_1 *standard deviation*.**

Wrap Up

- There are five primary threats to the internal validity of a multiple regression study:
  1. Omitted variables
  2. Functional form misspecification
  3. Errors in variables (measurement error in the regressors)
  4. Missing Data and Sample selection
  5. Simultaneous causality
- Besides, the data structure may violate the 2th OLS regression assumption, thus random sampling.
  1. Times series
  2. Cluster data
  3. Spatial data
- Last but not least, the **magnitude of** β_1 matters.

Wrap Up

- Each of these, if present, results in failure of the first least squares assumption,which in turn means that the OLS estimator is biased and inconsistent.
- Incorrect calculation of the standard errors also poses a threat to internal validity.
- Applying this list of threats to a multiple regression study provides a systematic way to assess the internal validity of that study.

External validity

Definition

- Suppose we estimate a regression model that is internally valid.
- Can the statistical inferences be generalized from the population and setting studied to other populations and settings?

Threats to external validity

1. Differences in populations
    - The population from which the sample is drawn might differ from the population of interest
    - For example, if you estimate the returns to education for *men*, these results might not be informative if you want to know the returns to education for *women*.
2. Differences in settings
    - The setting studied might differ from the setting of interest due to differences in laws, institutional environment and physical environment.
    - For example, the estimated returns to education using data from the U.S might not be informative for China.
    - Because the educational system is different and different institutions of the labor market.

Application to the case of class size and test score

- This analysis was based on test results for California school districts.
- Suppose for the moment that these results are internally valid. To what other populations and settings of interest could this finding be generalized?
  - generalize to colleges: it is implausible
  - generalize to other U.S. elementary school districts: it is plausible

Wrap up

- It is not easy to make your studies valid internally.
- Even harder when you consider generalize your findings.
- Then common way to generalize the findings actually is to repeat to make the studies internal valid.
- Then we make a generalizing conclusions based on a bunch of internal valid studies.

Example: Test Scores and Class Size

External Validity

- Whether the California analysis can be generalized—that is, whether it is externally valid—depends on the population and setting to which the generalization is made.
- we consider whether the results can be generalized to other elementary public school districts in the United States.
  - more specifically, 220 public school districts in *Massachusetts* in 1998.
  - if we find similar results in the California and Massachusetts, it would be evidence of external validity of the findings in California.
  - Conversely, finding different results in the two states would raise questions about the internal or external validity of at least one of the studies.

Comparison of the California and Massachusetts data.

TABLE 9.1 Summary Statistics for California and Massachusetts Test Score Data Sets

	California		Massachusetts	
	Average	Standard Deviation	Average	Standard Deviation
Test scores	654.1	19.1	709.8	15.1
Student–teacher ratio	19.6	1.9	17.3	2.3
% English learners	15.8%	18.3%	1.1%	2.9%
% Receiving lunch subsidy	44.7%	27.1%	15.3%	15.1%
Average district income ($)	$15,317	$7226	$18,747	$5808
Number of observations	420		220	
Year	1999		1998	

Regressor	(1)	(2)	(3)	(4)	(5)	(6)
Student–teacher ratio (*STR*)	−1.72** (0.50)	−0.69* (0.27)	−0.64* (0.27)	12.4 (14.0)	−1.02** (0.37)	−0.67* (0.27)
STR^2				−0.680 (0.737)		
STR^3				0.011 (0.013)		
% English learners		−0.411 (0.306)	−0.437 (0.303)	−0.434 (0.300)		
% English learners > median? (Binary, *HiEL*)					−12.6 (9.8)	
HiEL × *STR*					0.80 (0.56)	
% Eligible for free lunch		−0.521** (0.077)	−0.582** (0.097)	−0.587** (0.104)	−0.709** (0.091)	−0.653** (0.72)
District income (logarithm)		16.53** (3.15)				
District income			−3.07 (2.35)	−3.38 (2.49)	−3.87* (2.49)	−3.22 (2.31)
District income^2			0.164 (0.085)	0.174 (0.089)	0.184* (0.090)	0.165 (0.085)
District income^3			−0.0022* (0.0010)	−0.0023* (0.0010)	−0.0023* (0.0010)	−0.0022* (0.0010)
Intercept	739.6** (8.6)	682.4** (11.5)	744.0** (21.3)	665.5** (81.3)	759.9** (23.2)	747.4** (20.3)

Test scores and class size in MA

F-Statistics and *p*-Values Testing Exclusion of Groups of Variables

	(1)	(2)	(3)	(4)	(5)	(6)
All *STR* variables and interactions = 0				2.86 (0.038)	4.01 (0.020)	
$STR^2, STR^3 = 0$				0.45 (0.641)		
$Income^2, Income^3$			7.74 (< 0.001)	7.75 (< 0.001)	5.85 (0.003)	6.55 (0.002)
$HiEL, HiEL \times STR$					1.58 (0.208)	
SER	14.64	8.69	8.61	8.63	8.62	8.64
$\overline{R}^2$	0.063	0.670	0.676	0.675	0.675	0.674

These regressions were estimated using the data on Massachusetts elementary school districts described in Appendix 9.1. Standard errors are given in parentheses under the coefficients, and *p*-values are given in parentheses under the *F*-statistics. Individual coefficients are statistically significant at the *5% level or **1% level.

Test scores and class size in MA

TABLE 9.3 Student–Teacher Ratios and Test Scores: Comparing the Estimates from California and Massachusetts

	OLS Estimate $\hat{\beta}_{STR}$	Standard Deviation of Test Scores Across Districts	Estimated Effect of Two Fewer Students per Teacher, In Units of: Points on the Test	Standard Deviations
California				
Linear: Table 9.3(2)	−0.73 (0.26)	19.1	1.46 (0.52)	0.076 (0.027)
Cubic: Table 9.3(7) *Reduce STR from 20 to 18*	—	19.1	2.93 (0.70)	0.153 (0.037)
Cubic: Table 9.3(7) *Reduce STR from 22 to 20*	—	19.1	1.90 (0.69)	0.099 (0.036)
Massachusetts				
Linear: Table 9.2(3)	−0.64 (0.27)	15.1	1.28 (0.54)	0.085 (0.036)

Standard errors are given in parentheses.

Internal Validity

- The similarity of the results for California and Massachusetts does not ensure their internal validity.
- **Omitted variables**: teacher quality or a low student-teacher ratio might have families that are more committed to enhancing their children's learning at home or migrating to a better district.
- **Functional form**: Although further functional form analysis could be carried out, this suggests that the main findings of these studies are unlikely to be sensitive to using different nonlinear regression specifications.
- **Errors in variables**: The average student-teacher ratio in the district is a broad and potentially inaccurate measure of class size.
  - Because students' mobility, the STR might not accurately represent the actual class sizes, which in turn could lead to the estimated class size effect being biased toward zero.

Internal Validity

- **Selection**: data cover all the public elementary school districts in the state that satisfy minimum size restrictions, so there is no reason to believe that sample selection is a problem here.
- **Simultaneous causality**: it would arise if the performance on tests affected the student–teacher ratio.
- **Heteroskedasticity** and **correlation of the error term** across observations.
  - It does not threaten internal validity.
  - Correlation of the error term across observations, however, could threaten the consistency of the standard errors because the assumption of simple random sampling is violated.

Appendix

Math Review: Truncated Density Function

Truncated Density Function

The proof follows from the definition of a conditional probability is

$$Pr(A|B) = \frac{Pr(AB)}{Pr(B)}$$

then,

$$F(x|X > c) =$$

Math Review: Truncated Density Function

Truncated Density Function

The proof follows from the definition of a conditional probability is

$$Pr(A|B) = \frac{Pr(AB)}{Pr(B)}$$

then,

$$F(x|X > c) = \frac{Pr(X < x, X > c)}{Pr(X > c)}$$

Math Review: Truncated Density Function

Truncated Density Function

The proof follows from the definition of a conditional probability is

$$Pr(A|B) = \frac{Pr(AB)}{Pr(B)}$$

then,

$$F(x|X > c) = \frac{Pr(X < x, X > c)}{Pr(X > c)} = \frac{Pr(c < X < x)}{1 - F(c)}$$

Math Review: Truncated Density Function

Truncated Density Function

The proof follows from the definition of a conditional probability is

$$Pr(A|B) = \frac{Pr(AB)}{Pr(B)}$$

then,

$$F(x|X > c) = \frac{Pr(X < x, X > c)}{Pr(X > c)} = \frac{Pr(c < X < x)}{1 - F(c)}$$
$$= \frac{F(x) - F(c)}{1 - F(c)}$$

then,

$$f(x|x > c) = \frac{d}{dx}F(x|X > c) = \frac{\frac{d}{dx}[F(x)] - 0}{1 - F(c)} = \frac{f(x)}{1 - F(c)}$$

The Expectation in a Standard Normal Truncated

Proof

$$E(x|x > c) =$$

The Expectation in a Standard Normal Truncated

Proof

$$E(x|x > c) = \int_c^{+\infty} xf(x|x > c)dx =$$

The Expectation in a Standard Normal Truncated

Proof

$$E(x|x > c) = \int_c^{+\infty} xf(x|x > c)dx = \int_c^{+\infty} x\frac{\phi(x)}{1 - \Phi(c)}dx$$

The Expectation in a Standard Normal Truncated

Proof

$$E(x|x>c) = \int_c^{+\infty} x f(x|x>c)dx = \int_c^{+\infty} x \frac{\phi(x)}{1-\Phi(c)}dx$$

$$= \frac{1}{1-\Phi(c)} \int_c^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

The Expectation in a Standard Normal Truncated

Proof

$$
\begin{aligned}
E(x|x>c) &= \int_c^{+\infty} x f(x|x>c)dx = \int_c^{+\infty} x\frac{\phi(x)}{1-\Phi(c)}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}d(\frac{x^2}{2})
\end{aligned}
$$

The Expectation in a Standard Normal Truncated

Proof

$$
\begin{aligned}
E(x|x>c) &= \int_c^{+\infty} x f(x|x>c)dx = \int_c^{+\infty} x\frac{\phi(x)}{1-\Phi(c)}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}d(\frac{x^2}{2}) \\
&= \frac{1}{1-\Phi(c)}\int_{\frac{c^2}{2}}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-t}d(t)
\end{aligned}
$$

The Expectation in a Standard Normal Truncated

Proof

$$
\begin{aligned}
E(x|x>c) &= \int_c^{+\infty} x f(x|x>c)dx = \int_c^{+\infty} x\frac{\phi(x)}{1-\Phi(c)}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}d(\frac{x^2}{2}) \\
&= \frac{1}{1-\Phi(c)}\int_{\frac{c^2}{2}}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-t}d(t) \\
&= \frac{1}{1-\Phi(c)} \times \frac{1}{\sqrt{2\pi}}\left[-e^{-t}\right]_{\frac{c^2}{2}}^{+\infty}
\end{aligned}
$$

The Expectation in a Standard Normal Truncated

Proof

$$
\begin{aligned}
E(x|x>c) &= \int_c^{+\infty} x f(x|x>c)dx = \int_c^{+\infty} x\frac{\phi(x)}{1-\Phi(c)}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} x\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{1-\Phi(c)}\int_c^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}d(\frac{x^2}{2}) \\
&= \frac{1}{1-\Phi(c)}\int_{\frac{c^2}{2}}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-t}d(t) \\
&= \frac{1}{1-\Phi(c)}\times\frac{1}{\sqrt{2\pi}}\left[-e^{-t}\right]_{\frac{c^2}{2}}^{+\infty} \\
&= \frac{1}{1-\Phi(c)}\times\frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}} = \frac{\phi(c)}{1-\Phi(c)}
\end{aligned}
$$