Lecture 6: Binary Dependent Variable

Introduction to Econometrics,Spring 2026

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Review of the last lecture

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- What if the outcome variables(Y) is **discrete** or **limited**.

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- Interpreting the results more difficult for the nonlinearity.

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- **Binary outcomes models** and **Multinomial choice models** are covered here.

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The Linear Probability Model(LPM)

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- Then we can extend it to the **conditional expectation** of Y equals to the the probability of $Y = 1$ conditional on Xs,thus

$$E[Y|X_{1i}, ..., X_{ki}] = Pr(Y = 1|X_{1i}, ..., X_{ki})$$

Multiple OLS Regression

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- Now a **Linear Probability Model** can be defined as following

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- β_j can be explained as **the change in the probability** that $Y = 1$ associated with **a unit change in** X_j

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    - One difference is that both original R^2 and adjusted-R^2 are not meaningful statistics now.

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Variable	Description	Mean	SD
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- **The estimated coefficient is significantly different from 0 at a 1% significance level as the t-statistic is over 6.**

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- **Question**: Does the effect matter? Or the magnitude of the effect is economically large enough.
- **Answer**: An option is comparing with the **mean** of dependent variable.
  - Here $deny\ rate = 0.12$ means that the deny ratio will increase $0.06/0.12 \times 100\% = 50\%$ if P/I Ratio increases 10%.

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- **This coefficient is significant at the 1% level (the t-statistic is 7.11).**

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- Advantages of the linear probability model:
    - Easy to estimate and inference
    - Coefficient estimates are easy to interpret
    - Very useful under some circumstances like using IV.

LPM's Weakness: Heteroskedasticity

- The conditional variance of the error term u_i is always heteroskedasticity.

$$\text{Var}\left(u_i \mid X_{1i}, \cdots, X_{ki}\right) \neq \sigma_u^2$$

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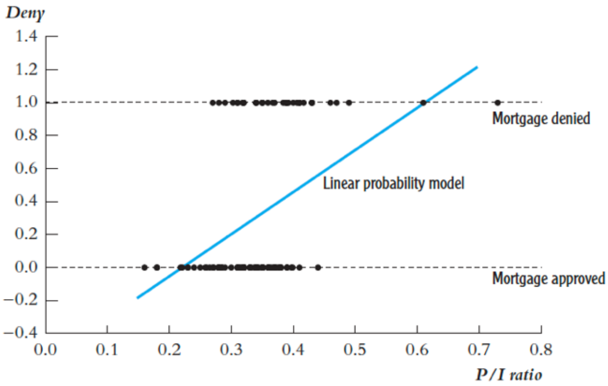
- Always use **heteroskedasticity robust standard errors** when estimating a linear probability model!

LPM’s Weakness: Predicted values

- More serious problem: **the predicted probability can be below 0 or above 1!**

FIGURE 11.1 Scatterplot of Mortgage Application Denial and the Payment-to-Income Ratio

Mortgage applicants with a high ratio of debt payments to income (*P/I ratio*) are more likely to have their application denied (*deny* = 1 if denied, *deny* = 0 if approved). The linear probability model uses a straight line to model the probability of denial, conditional on the *P/I ratio*.



Nonlinear Probability Models

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&= G(\beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i})
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where $Z = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i}$

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- **Probit regression models the probability that $Y = 1$**

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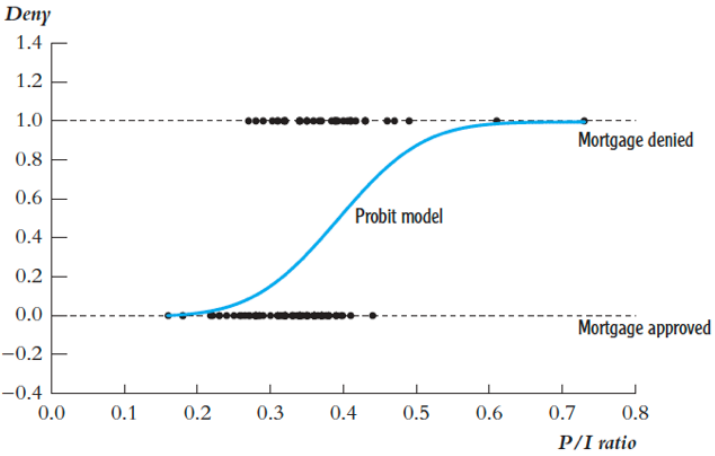
- Then it make sure that the **predicted probabilities** of the probit model are between 0 and 1.

Probit Model: Shape and Prediction Value

FIGURE 11.2 Probit Model of the Probability of Denial, Given *P/I Ratio*

The probit model uses the cumulative normal distribution function to model the probability of denial given the payment-to-income ratio or, more generally, to model $\Pr(Y = 1 \mid X)$. Unlike the linear probability model, the probit conditional probabilities are always between 0 and 1.



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  1. computing the predicted probability for the initial value of the regressors,
  2. computing the predicted probability for the new or changed value of the regressors,
  3. taking their difference.

Probit Model: Explaination to the Coefficient

The Expected Change on Y of a Change in X_1 in the Nonlinear Regression Model (8.3)

KEY CONCEPT 8.1

The expected change in Y, ΔY, associated with the change in X_1, ΔX_1, holding $X_2, \ldots, X_k$ constant, is the difference between the value of the population regression function before and after changing X_1, holding $X_2, \ldots, X_k$ constant. That is, the expected change in Y is the difference:

$$\Delta Y = f(X_1 + \Delta X_1, X_2, \ldots, X_k) - f(X_1, X_2, \ldots, X_k). \qquad (8.4)$$

The estimator of this unknown population difference is the difference between the predicted values for these two cases. Let $\hat{f}(X_1, X_2, \ldots, X_k)$ be the predicted value of Y based on the estimator $\hat{f}$ of the population regression function. Then the predicted change in Y is

$$\Delta \hat{Y} = \hat{f}(X_1 + \Delta X_1, X_2, \ldots, X_k) - \hat{f}(X_1, X_2, \ldots, X_k). \qquad (8.5)$$

The Predicted Probability: one regressor

- **Suppose the probit population regression model with only one regressors, X_1**

$$Pr(Y = 1|X_1) = \Phi(Z) = \Phi(\beta_0 + \beta_1 X_1)$$

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$$Z = -2 + 3X_1$$

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- **Question**: *how to compute the probability change of X_1 with a change from 0.4 to 0.5?*

The Predicted Probability: one regressor

- The probability that $Y = 1$ when $X_1 = 0.4$, then $z = -2 + 3 \times 0.4 = -0.8$, then the predicted probability is

$$Pr(Y = 1|X_1 = 0.4) = Pr(z \leq -0.8) = \Phi(-0.8)$$

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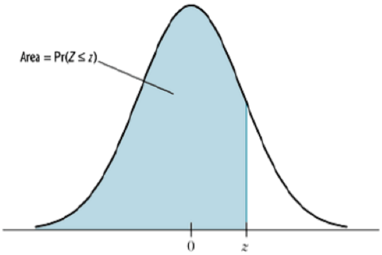
- Then the difference is

$$Pr(Y = 1|X_1 = 0.5) - Pr(Y = 1|X_1 = 0.4) =$$
$$\Phi(-.5) - \Phi(-.8) = 0.3085 - 0.2119 = 0.097$$

The Predicted Probability: one regressor

TABLE 1 The Cumulative Standard Normal Distribution Function, $\Phi(z) = \Pr(Z \text{ " } z)$



z	Second Decimal Value of z									
	0	1	2	3	4	5	6	7	8	9
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121

Example: Mortgage Applications

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- **Answer**: Yes, the payment to income ratio affects whether or not a mortgage application is denied.

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- **Answer**: The estimated change in the probability of denial is $0.159 - 0.097 = 0.062$, which means that the P/I ratio increase from *from 0.3 to 0.4*, the denial probability increase 6.2%.

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- Hence, the effect of a change in X depends on the starting value of X and other Xs like other nonlinear functions.

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  - **Average Marginal Effect** (AME): average of ME at each $X = X_i$ (at sample values and then average)

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  - **Marginal Effect at a Representative Value** (MER):ME at $X = X^*$ (at representative values of the regressors)
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  - **Average Marginal Effect** (AME): average of ME at each $X = X_i$ (at sample values and then average)
- The most common one is MEM while the other two are not meaningless.

Example: Mortgage Applications

- The **Marginal Effect**

$$\frac{\partial Pr(deny = 1|P/I\ ratio)}{\partial P/I\ ratio} = \phi(-2.19 + 2.97P/I\ ratio) \times 2.97$$

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- **The the effect of** $P/I\ ratio$ **change** 10%**(0.1) on the probability of deny is** $3.36\%(0.0336)$

Effect of a Change in X: When X is discrete

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- Assume X_2 is a dummy variable, then partial effect of X_2 changing from 0 to 1:

$$G(\beta_0 + \beta_1 X_{1,i} + \beta_2 \times 1 + ... + \beta_k X_{k,i}) - G(\beta_0 + \beta_1 X_{1,i} + \beta_2 \times 0 + ... + \beta_k X_{k,i})$$

Example: Race in Mortgage Applications

- **Mortgage denial (deny) and the payment-income ratio (P/I ratio) and race**

$$\widehat{Pr(deny = 1|P}/I\ ratio) = \Phi(\underset{(0.16)}{-2.26} + \underset{(0.44)}{2.74}P/I\ ratio + \underset{(0.083)}{0.71}black)$$

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- **Answer**: The difference between whites and blacks at $P/Iratio = 0.3$ is $0.233 - 0.075 = 0.158$, which means probability of denial for blacks is 15.8% higher

Logit Model

Logistic Function

- Using the standard **logistic** cumulative distribution function

$$
\begin{aligned}
Pr(Y_i = 1|Z) &= \frac{1}{1+e^{-Z}} \\
&= \frac{e^{Z}}{1+e^{Z}}
\end{aligned}
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- Since $F(z) = Pr(Z \leq z)$ we have that the predicted probabilities of the logit model are also between 0 and 1.

Logit Model: Predicted Probabilities

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- So the probability is

$$\begin{aligned} Pr(Y = 1 | X_1 = 0.4) = Pr(Z \leq -0.8) \\ =& F(-0.8) \\ =& \frac{1}{1 + e^{-0.8}} \\ =& 0.31 \end{aligned}$$

Logit Model: Predicted Probabilities

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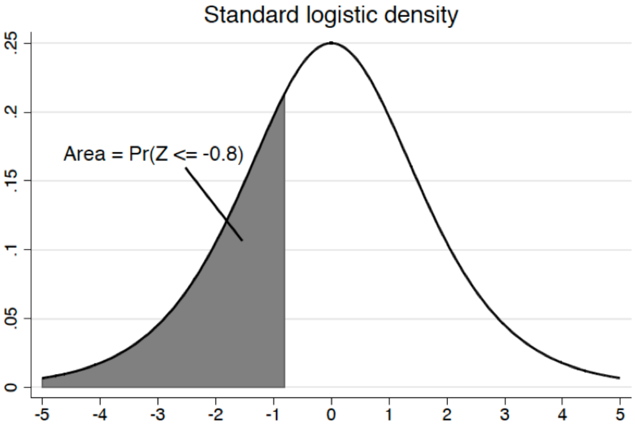

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  - The **odds ratio**

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- Then the **ratio** of probability of $Y = 1$ to the probability of $Y = 0$ is

$$\frac{p}{1 - p} = \frac{Pr(Y_i = 1|Z)}{Pr(Y_i = 0|Z)} = e^z$$

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- the $\frac{p}{1-p}$ is called as **Odds Ratio** .

Logit Model: the Odds Ratio

- Then the logit model can be expressed as

$$ln(\frac{p}{1-p}) = Z = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + ... + \beta_k X_{k,i}$$

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- Therefore $100 \times \hat{\beta}_j$ can be expressed that the **percentage change in odds ratio** arising from 1 unit change in X_j.

Example: Mortgage Applications

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$$\widehat{Pr(deny = 1|P/I\ ratio)} = F(\underset{(0.359)}{-4.03} + \underset{(1.000)}{5.88}P/I\ ratio)$$

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- **If** $P/I\ ratio$ **increases** 10%**(0.1), then odds ratio of deny to accept will be increased** 58.8%**.**

Marginal Effect in logit model

- Then **Marginal Effect at Mean** (MEM):(at the sample mean of the regressors:
  $P/I\ ratio_{mean} = 0.331$

$$\frac{\partial Pr(deny = 1|P/I\ ratio)}{\partial P/I\ ratio}_{at\ mean} = f(-2.19 + 2.97 \times 0.331) \times 2.97$$
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- **The the effect of $P/I\ ratio$ change 10%(0.1) on the probability of deny is** $5.26\%(0.0526)$

Example: Mortgage Applications on Race

- **Logit Model: Mortgage denial (deny) and the payment-to-income ratio (P/I ratio) and race**

$$Pr(deny \widehat{= 1|P}/I\ ratio) = F(\underset{(0.35)}{-4.13} + \underset{(0.96)}{5.37} P/I\ ratio + \underset{(0.15)}{1.27} black)$$

Example: Mortgage Applications on Race

- **The predicted denial probability of a *white* applicant with $P/I\ ratio = 0.3$ is**

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$$\frac{1}{1+e^{-(-4.13+5.37\times 0.3+1.27\times 1)}} = 0.222$$

- the difference is

$$0.222 - 0.074 = 0.148 = 14.8\%$$

which indicates that the probability of denial for blacks is 14.8% higher than that for whites when $P/Iratio = 0.3$.

Maximum Likelihood Estimation(MLE) to Probit and Logit

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- **How do we estimate $\beta_0, \beta_1, ..., \beta_k$?**

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- Logit and Probit models are nonlinear in the coefficients $\beta_0, \beta_1, ..., \beta_k$
  - These models cannot be estimated directly by OLS, but require **Nonlinear Least Squares (NLS)**.
  - In practice, **Maximum Likelihood Estimation** (MLE) is the most common method for estimating logit and probit models.

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- **MLE's logic**: the most likely function is the function to have produce the data we observed.

Review: Maximum Likelihood Estimation

- **Random Variables Y_i have n observations, thus $Y_1, Y_2, Y_3, ...Y_n$ have a joint density function** denoted

$$f_\theta(Y_1, Y_2, ..., Y_n) = f(Y_1, Y_2, ..., Y_n|\theta)$$

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  - **where θ is an unknown parameter.**
- **Given observed values $Y_1 = y_1, Y_2 = y_2, ..., Y_n = y_n$, the likelihood of θ is the function**

$$likelihood(\theta) = f(Y_1 = y_1, Y_2 = y_2, ..., Y_n = y_n|\theta) = f(\theta; y_1, ..., y_n)$$

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  - which can be considered as a function of θ.
- Then the **Maximum Likelihood Estimation** to θ is a solution to the question

$$arg\max_{\hat{\theta}} f(\theta; Y_1 = y_1, ..., Y_n = y_n))$$

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$$Y_i = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1 - p \end{cases}$$

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- Which can be transform into a **probability density function** as

$$Pr(Y_i = y) = Pr(Y_i = 1)^y(1 - Pr(Y_i = 1))^{1-y} = p^y(1-p)^{1-y}$$

Maximum Likelihood Estimation of a Binary Variable

MLE Step 1: *write down the likelihood function, the joint probability distribution of the data*

- Since $Y_1, ..., Y_n$ are **i.i.d**,the joint probability distribution of the observations, thus the Likelihood function is the **product** of the individual distributions

$$f_{bernouilli}(p; Y_1 = y_1, ..., Y_n = y_n) = Pr(Y_1 = y_1, ..., Y_n = y_n)$$

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- Then the **maximization** problem is

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- Then the MLE estimator for a binary variable, p, is $\hat{p}_{MLE} = \frac{1}{n}\sum y_i = \overline{Y}$

MLE of the Probit Model

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$$= \left[\Phi(\beta_0 + \beta_1 X_{11} + ... + \beta_k X_{k1})^{y_1}(1 - \Phi(\beta_0 + \beta_1 X_{11} + ... + \beta_k X_{k1}))^{1-y_1}\right] \times$$

$$... \times \left[\Phi(\beta_0 + \beta_1 X_{1n} + ... + \beta_k X_{kn})^{y_n}(1 - \Phi(\beta_0 + \beta_1 X_{1n} + ... + \beta_k X_{kn}))^{1-y_n}\right]$$

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- Similar to the Probit model but with a different function for p_i

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- We can still obtain the values of estimators using **numerical algorithm** with iterative methods.
- One of common methods is **Newton-Raphson Method** based on low order *Taylor series expansions*.

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1. *fraction correctly predicted*
    - If $Y_i = 1$ and the predicted probability exceeds 50% or if $Y_i = 0$ and the predicted probability is less than 50%, then Y_i is said to be correctly predicted.

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Statistical inference based on the MLE

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    - For example, the **95% confidence intervals** are formed as 1.96 standard errors.

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  - Therefore, the likelihood ratio test statistic is always non-negative.

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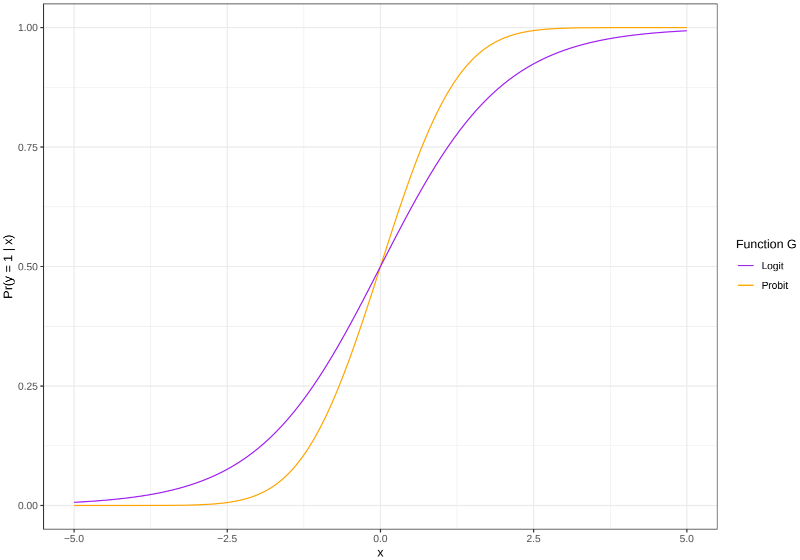
- So which should you use in practice?
  - *There is no one right answer*, and different researchers use different models.
  - *Probit and logit regressions frequently produce similar results*.

Logit v.s. Probit



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- Coefficients can be compared across models, using the following rough conversion factors (Amemiya 1981)

$$\hat{\beta}_{logit} \simeq 4\hat{\beta}_{ols}$$
$$\hat{\beta}_{probit} \simeq 2.5\hat{\beta}_{ols}$$
$$\hat{\beta}_{logit} \simeq 1.6\hat{\beta}_{probit}$$

Example: Mortgage Applications(short regression)

Dependent variable: *deny* = 1 if mortgage application is denied, = 0 if accepted			
regression model	LPM	Probit	Logit
black	0.177***	0.71***	1.27***
	(0.025)	(0.083)	(0.15)
P/I ratio	0.559***	2.74***	5.37***
	(0.089)	(0.44)	(0.96)
constant	-0.091***	-2.26***	-4.13***
	(0.029)	(0.16)	(0.35)
difference Pr(*deny*=1) between black and white applicant when *P/I ratio*=0.3	17.7%	15.8%	14.8%

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- One important feature: the outcomes cannot be ordered in any natural way.

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- **Using MLE estimation to maximize the log-likelihood function to solve the parameters β.**

$$\ln L(\cdot) = \ln\left(\prod_{i=1}^{N} f_{ij}\,(\beta : X)\right) = \ln\left(\prod_{i=1}^{N}\prod_{j=1}^{m} p_j^{y_j}\right) = \sum_{i=1}^{N}\sum_{j=1}^{m} y_{ij}\ln p_{ij}$$

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  - Which is the necessary condition for the identification of the model.

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$$P(Y_i = J | X_{1i}, ..., X_{ki}) = \begin{cases} \frac{1}{1+\sum_{j=2}^{M} exp(X'\beta^j)} & \textbf{if J=1} \\ \frac{exp(X'\beta^J)}{1+\sum_{j=2}^{M} exp(X'\beta^j)} & \textbf{if J=2,3,...,M} \end{cases}$$

Multinomial Logit Model: Coefficients Interpretation

- Then *the probability that the outcome is alternative* J can be expressed as following under the distributional assumption of the error term:(Skip the derivation, you can prove it by yourself.)

$$p_{ij} = P(Y_i = J|X_{1i}, ..., X_{ki}) = \frac{exp(X'\beta^J)}{\Sigma_{j=1}^{M} exp(X'\beta^j)}$$

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- **Answer**:the parameters of the multinomial logit model are difficult to interpret. Neither the sign nor the magnitude of the parameter has an direct intuitive meaning.

Multinomial Logit Model: Marginal Effects

- The **marginal probability effects** of the multinomial logit model for a change of X_k for choice J can be calculated as follows:

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$$\widehat{MPEM}_{jk} = \bar{p}_{ij}\left(\hat{\beta}_{jk} - \sum_{j=1}^{M}\bar{p}_{ij}\right) \quad (j = 1, \ldots, M)$$

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- Recall the **odds ratio** in the binary choice model, thus the **ratio of probability** of $Y = 1$ to the probability of $Y = 0$ is

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- Therefore $100 \times \hat{\beta}_k$ can be expressed that **the percentage change in odds ratio for choice J relative to the base category** 1 arising from a unit change in X_k.

Multinomial Logit Model: Strong Assumption

- Likewise,the odds between two alternatives j and k is

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- Essentially, the IIA assumption requires that all the alternatives are **independent of each other**.

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  - This is because subway and bus are **closer substitutes** than car and bus.
- Therefore, we have more flexible models to relax the IIA assumption as **nested logit model** and **mixed logit model**.(You may learn them in some advanced courses in your future study.)

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  - The ordered probit or logit models are used when the dependent variable is ordinal.

A Lastest Application: Jia,Lan and Miquel(2021)

Parental background and Entrepreneurship in China

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Jia,Lan and Miquel(2021): Data

1. Individual-level data:
    - China General Social Survey (GCSS) 2006,2008,2010,2012,2013
    - 31 provinces, 22801 urban respondents.
2. Province-level data:
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Jia,Lan and Miquel(2021): Main Variables

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  - **self-employment**: owners of non-incorporated small businesses.
  - **goverment employee**: work in government or in a public organization affiliated with the government.

Parental Background and Doing Business

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  - $Prov_p \times Year_t$ are the province-by-year fixed effects.

Empirical Results: LPM

Table 3A
Parent background and child occupations: OLS estimates.

	(1)	(2)	(3)	(4)	(5)	(6)
	Government worker (0/1, mean = 0.217)		Business owner (0/1, mean = 0.022)		Self-employed (0/1, mean = 0.107)	
Cadre Parent	0.144***	0.115***	0.006**	0.003	−0.009*	−0.011**
	(0.009)	(0.009)	(0.003)	(0.003)	(0.005)	(0.005)
Entrepreneur Parent	−0.006	−0.006	0.016***	0.014**	0.063***	0.057***
	(0.012)	(0.011)	(0.006)	(0.006)	(0.013)	(0.013)
Province FE*Year FE	Y	Y	Y	Y	Y	Y
Individual Characteristics		Y		Y		Y
Observations	22,801	22,801	22,801	22,801	22,801	22,801
R-squared	0.057	0.139	0.015	0.022	0.039	0.067

- **Cadre Parents** increase the probability of being government workers(11.5%).

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- **Cadre Parents** increase the probability of being government workers(11.5%).
- **Entrepreneur Parents** do not.

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- **Entrepreneur Parents** increase the probability of being business owner(1.6%).
- **Cadre Parents** also increase the probability of being business owner(0.6%). However, the effect will go away when controlling individual characteristics.

Empirical Results: LPM

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Parent background and child occupations: OLS estimates.

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Province FE*Year FE	Y	Y	Y	Y	Y	Y
Individual Characteristics		Y		Y		Y
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- **Entrepreneur Parents** increase the probability of being self-employed(6%).

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Province FE*Year FE	Y	Y	Y	Y	Y	Y
Individual Characteristics		Y		Y		Y
Observations	22,801	22,801	22,801	22,801	22,801	22,801
R-squared	0.057	0.139	0.015	0.022	0.039	0.067

- **Entrepreneur Parents** increase the probability of being self-employed(6%).
- **Cadre Parents** *decrease* the probability of self-employment(1.1%).

Empirical Results: Multinomial Logit

Table 3B
Relative risk ratios in diff. Occupations by parental background –multinomial logit estimates.

	(1)	(2)	(3)
Reference group: being a firm employee			
Work in government			
Cadre Parents	2.327***	2.056***	2.043***
	(0.122)	(0.120)	(0.104)
Entrepreneur Parents	1.116	1.047	1.047
	(0.095)	(0.093)	(0.092)
Being a business owner			
Cadre Parents	1.656***	1.406***	1.413***
	(0.187)	(0.162)	(0.167)
Entrepreneur Parents	2.225***	1.912***	1.750***
	(0.379)	(0.315)	(0.300)
Being self-employed			
Cadre Parents	1.120*	1.058	1.080
	(0.075)	(0.070)	(0.073)
Entrepreneur Parents	1.921***	1.763***	1.579***
	(0.186)	(0.169)	(0.015)
Individual Characteristics		Y	Y
Province FE*Year FE			Y
Observations	22,801	22,801	22,801

Notes: In Table 3A, the comparison group is all other occupations. In Table 3B, the reference group is being a firm employee. Individual characteristics include: age, gender, marital status, ethnic minority status and college education. Standard errors are clustered at the province-year level. Significance level: *$p < 0.1$, **$p < 0.05$, ***$p < 0.01$.

Work in government

- **Cadre Parents** increase the odds of being a government relative to be a firm employee by **over 2 times significantly**.
- **Entrepreneur Parents** increase the odds of being a government relative to be a firm employee but **the effect is not significant**.

Empirical Results: Multinomial Logit

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Being a business owner

- **Cadre Parents** increase the odds of being a business owner relative to be a firm employee by **over 1.4 times significantly**.
- **Entrepreneur Parents** increase the odds of being a business owner relative to be a firm employee by **over 1.7 times significantly**.

Empirical Results: Multinomial Logit

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Being self-employed

- **Cadre Parents** <u>don't increase the odds</u> of being self-employed relative to be a firm employee.
- **Entrepreneur Parents** <u>increase the odds</u> of self-employed relative to be a firm employee by **over 1.6 times significantly**.

Summary of LPM and Multinomial Logit

Parents	Model	Government	Business Owner	Self-employ
Cadre	LPM	$\uparrow$	$\uparrow$	$\downarrow$
Cadre	MLogit	$\uparrow$	$\uparrow$	–
Entrepreneur	LPM	–	$\uparrow$	$\uparrow$
Entrepreneur	MLogit	–	$\uparrow$	$\uparrow$

- The LPM and MLogit models provide very similar results.

Parental Background and Local Economic Context

- **Measurement**: *Provincial Government Expenditure on Business-related activities(PGEB)* as a measure of the role of government on the private business environment.

Parental Background and Local Economic Context

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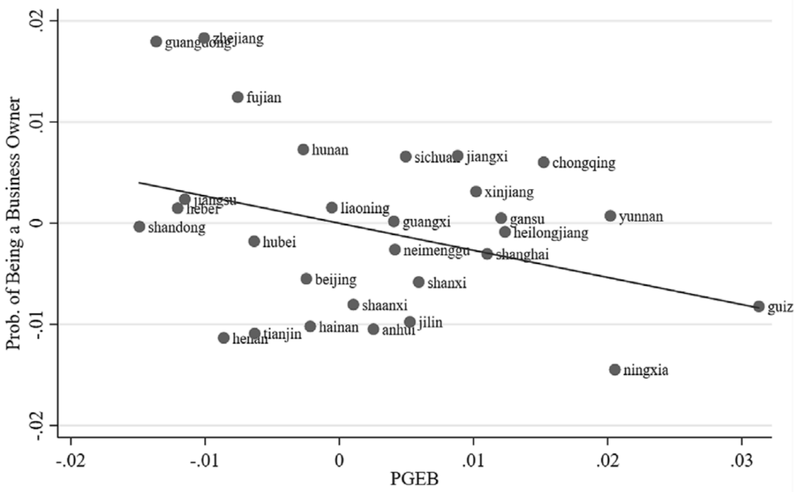
  - Expenditure on Business-related activities: Infrastructure and MCF (Manufacturing/Commerce/Finance).
- Robustness:
  - weakly correlated with GDP
  - negatively correlated marketization index.
  - relatively smaller share of private sector.

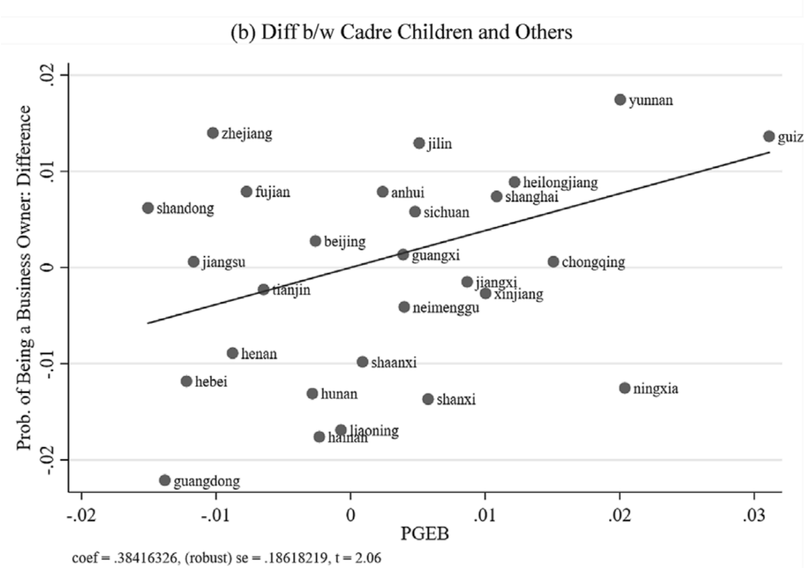
Descriptive patterns: cross-provinces

(a) Prob. Being a Business Owner



coef = -.26796817, (robust) se = .15185212, t = -1.76

Descriptive patterns: cross-provinces



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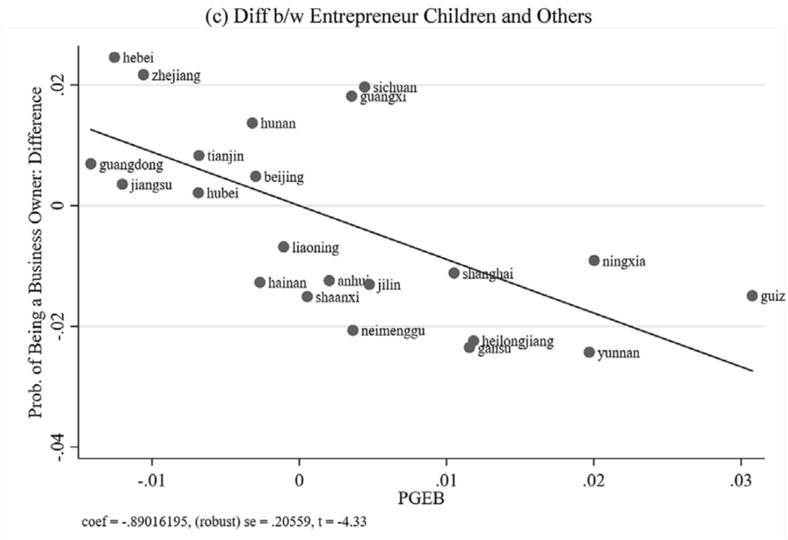

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- **Linear Probability Model: Interacted with PGEB**

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\begin{aligned}
Pr(Y=1|X) = \beta_1 CardreParent_i + \beta_2 CardreParent_i \times PGEB_{pt} \\
+\beta_3 EntreParents_i + \beta_4 EntreParents_i \times PGEB_{pt} \\
+\gamma X_i + \gamma X_i \times PGEB_{pt} + Prov_p \times Year_t + u_{ipt}
\end{aligned}
$$

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- **Question**: Which parameter is our interest? and how to interpret it?
- **Answer**: β_2 and β_4 are the coefficients of the interaction terms between parental occupation and PGEB.
- **Thinking 1**: *Why there is no PGEB term in the model?*

Empirical Results: LPM+Interactions

Table 4
The impact of cadre Parent × PGEB in determining business ownership.

	(1)	(2)	(3)	(4)	(5)	(6)
	Y = business owner (mean = 0.022)					
Cadre Parent * PGEB (sd)	0.004*	0.004*	0.005**			0.007**
	(0.002)	(0.002)	(0.002)			(0.003)
Cadre Parent	0.006**	0.003	0.003	0.003	0.003	0.003
	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)
Entrepreneur Parent * PGEB (sd)	−0.008*	−0.008**	−0.008*			−0.006
	(0.004)	(0.004)	(0.004)			(0.008)
Entrepreneur Parent	0.016***	0.014**	0.014**	0.014**	0.013**	0.014**
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)
Cadre Parent * GDP Per Capita (sd)				−0.001		−0.001
				(0.002)		(0.002)
Entre. Parent * GDP Per Capita (sd)				−0.006		−0.006
				(0.005)		(0.004)
Cadre Parent * Other Expend (sd)					0.003	−0.002
					(0.003)	(0.004)
Entrepreneur Parent * Other Expend (sd)					−0.007	−0.003
					(0.005)	(0.010)
Province FE*Year FE	Y	Y	Y	Y	Y	Y
Individual Characteristics		Y	Y	Y	Y	Y
PGEB *Individual Characteristics			Y	Y	Y	Y
Observations	22,801	22,801	22,801	22,801	22,801	22,801
R-squared	0.015	0.023	0.023	0.023	0.023	0.023

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Entre. Parent * GDP Per Capita (sd)				−0.006		−0.006
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Entrepreneur Parent * Other Expend (sd)					−0.007	−0.003
					(0.005)	(0.010)
Province FE*Year FE	Y	Y	Y	Y	Y	Y
Individual Characteristics		Y	Y	Y	Y	Y
PGEB *Individual Characteristics			Y	Y	Y	Y
Observations	22,801	22,801	22,801	22,801	22,801	22,801
R-squared	0.015	0.023	0.023	0.023	0.023	0.023

Empirical Results: LPM+Interactions

Table 4
The impact of cadre Parent × PGEB in determining business ownership.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Y = business owner (mean = 0.022)						Y = self-employed (mean = 0.107)
Cadre Parent * PGEB (sd)	0.004*	0.004*	0.005**			0.007**	0.002
	(0.002)	(0.002)	(0.002)			(0.003)	(0.007)
Cadre Parent	0.006**	0.003	0.003	0.003	0.003	0.003	−0.011**
	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.003)	(0.005)
Entrepreneur Parent * PGEB (sd)	−0.008*	−0.008**	−0.008*			−0.006	0.017
	(0.004)	(0.004)	(0.004)			(0.008)	(0.011)
Entrepreneur Parent	0.016***	0.014**	0.014**	0.014**	0.013**	0.014**	0.057***
	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.006)	(0.012)
Cadre Parent * GDP Per Capita (sd)				−0.001		−0.001	
				(0.002)		(0.002)	
Entre. Parent * GDP Per Capita (sd)				−0.006		−0.006	
				(0.005)		(0.004)	
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Province FE*Year FE	Y	Y	Y	Y	Y	Y	Y
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Observations	22,801	22,801	22,801	22,801	22,801	22,801	22,801
R-squared	0.015	0.023	0.023	0.023	0.023	0.023	0.068

Notes: This table shows that the advantage in becoming a business owner (1) increases with PGEB for those with cadre parents and (2) decreases with PGEB for those with entrepreneur parents. Individual characteristics include: age, gender, marital status, ethnic minority status, and college education. Standard errors are clustered at the province-year level. Significance level: $*p < 0.1$, $**p < 0.05$, $***p < 0.01$.

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   - Yes. the larger is government involvement in business-related spending, the larger the business-ownership propensity of children of government officials, and the smaller the propensity of children of entrepreneurs.

Wrap Up

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- Although Probit and Logit offer some advantages in model specifications over LPM, LPM is more intuitive and easier to interpret.
  - This is particularly useful when we want to deal with the endogeneity problem.
- When the dependent variable is binary,even multinomial, the LPM remains a good starting point for empirical analysis.

Appendix 1

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- If these assumptions fail, more specialized estimation methods may be needed.

Appendix 2: Newton-Raphson Method

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- We do not stop repeating this procedure until

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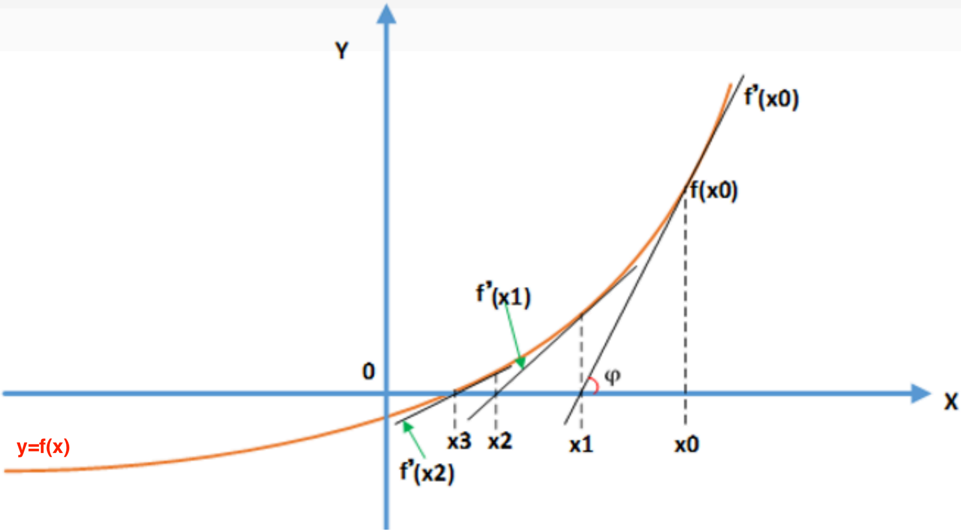

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