Introduction to Econometrics

Recite 2 : Review of Statistic Inference

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Outlines

Population, Parameters and Random Sampling

Population, Sample and i.i.d

- A **population** is a collection of people, items, or events about which you want to make inferences.
    - Population always have a probability distribution.
- *A* ***sample*** is a subset of population, which draw from population *in a certain way*.
- To represent the population well, a sample should be randomly collected and adequately large.
    - Infinite population
    - Finite population
        - With replacement
        - Without replacement: when the population size N is very large, compared with the sample size n, then we could say that they are *nearly independent.*

Random Sample and i.i.d

Definition

The r.v.s are called *a **random sample** of size n from the population* $f(x)$ if $X_1, ..., X_n$ are mutually independent and have the same p.d.f/p.m.f $f(x)$. Alternatively, $X_1, ..., X_n$ are called ***independent, and identically distributed*** *random variable with p.d.f/p.m.f , commonly abbreviated to i.i.d. r.v.s.*

- eg. Random sample of n respondents on a survey question.
- $X_i \perp X_j$ for all $i \neq j$
- $f_{X_i}(x)$ is the same for all i.
- And the joint p.d.f/p.m.f of $X_1, ..., X_n$ is given by

$$f(x_1, ..., x_n) = f(x_1)...f(x_n) = \prod_{i=1}^{n} f(x_i)$$

Statistic and Sampling Distribution

Definition

$X_1, ..., X_n$ is a *random sample* of size n from the population $f(x)$. A ***statistic*** is a real-valued or vector-valued function fully depended on $X_1, ..., X_n$, thus

$$T = T(X_1, ..., X_n)$$

- and the probability distribution of a statistic T is called the ***sampling distribution*** of T.
- A statistic is only a function of the sample.

Sample Mean and Sample Variance

Definition

The **sample average** or **sample mean**, $\overline{X}$, of the n observation $X_1, ..., X_n$ is

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + ... + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

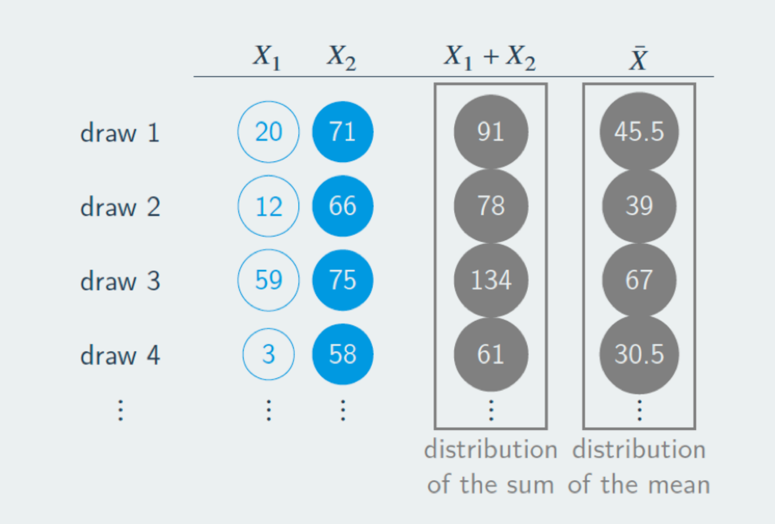
The **sample variance** is the statistic defined by

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

- if X_i is a r.v., then $\sum X_i$ is also a r.v.
- the sample mean and the sample variance are also a function of sums, so they are a r.v. too.
    - we could assume that the sample mean has some certain probability functions to describe its distributions.
    - what is the expectation, variance or p.d.f./c.d.f. of this distribution?

A simple case of sample mean

- Let $\{X_1, X_n\} \in [1, 100]$, assume $n = 2$, thus only $X_1$and X_2



Large-Sample Approximations to Sampling Distributions

Sampling Distributions

- There are two approaches to characterizing sampling distributions:
  - *exact/finite* sample distribution: The sampling distribution that exactly describes the distribution of X for any n is called the exact/finite sample distribution of $\overline{X}$.
  - *approximate/asymptotic* distribution: when the sample size n is large, the sample distribution approximates to a certain distribution function.
- Two key tools used to approximate sampling distributions when the sample size is large, assume that $n \to \infty$
  - The **Law of Large Numbers**(L.L.N.): when the sample size is large, $\overline{X}$ will be close to μ_Y , the population mean with very high probability.
  - The **Central Limit Theorem**(C.L.T.): when the sample size is large, the sampling distribution of the standardized sample average,$(\overline{Y}-\mu_Y)/\sigma_{\overline{Y}}$, is approximately normal.

Convergence in probability

Definition

Let $X_1, ..., X_n$ be an random variables or sequence, is said to converge in probability to a value b if for every $\varepsilon > 0$,

$$P(| X_n - b |> \varepsilon) \to 0$$

as $n \to \infty$. We write this $X_n \xrightarrow{p} b$ or $plim(X_n) = b$.

- it is similar to the concept of a limitation in a probability way.

the Law of Large Numbers

Theorem

Let $X_1, ..., X_n$ be an i.i.d draws from a distribution with mean μ and finite variance σ^2(a population) and $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean, then

$$\overline{X} \xrightarrow{p} \mu$$

- Intuition: the distribution of $\overline{X}_n$ "collapses" on μ.

A simple case

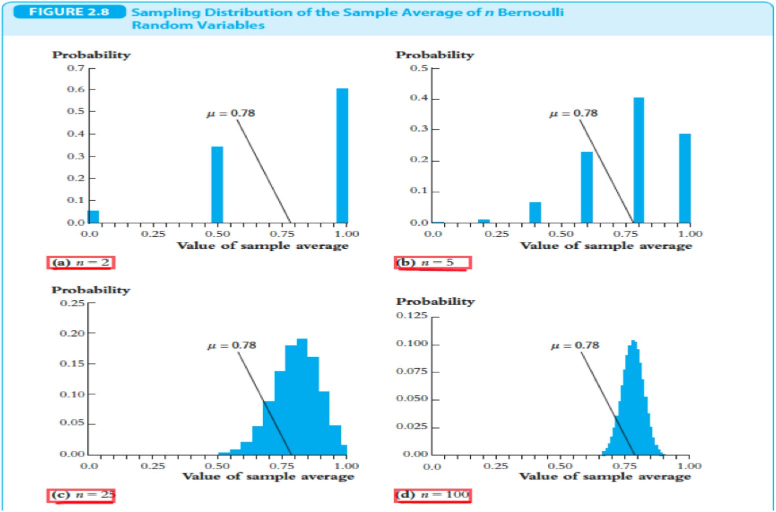
Example

Suppose X has a Bernoulli distribution if it have a binary values $X \in \{0, 1\}$ and its probability mass function is

$$P(X = x) = \begin{cases} 0.78 & if\, x = 1 \\ 0.22 & if\, x = 0 \end{cases}$$

- then $E(X) = p = 0.78$ and $Var(X) = p(1 - p) = 0.1716$.

Convergence in Distribution



Convergence in Distribution

Definition

Let $X_1, X_2, \ldots$ be a sequence of r.v.s, and for $n = 1, 2, \ldots$ let $F_n(x)$ be the c.d.f of X_n. Then it is said that $X_1, X_2, \ldots$ converges in distribution to r.v. W with c.d.f, F_W if

$$\lim_{n\to\infty} F_n(x) = F_W(x)$$

which we write as $X_n \xrightarrow{d} W$.

- Basically: when n is big, the distribution of X_n is very similar to the distribution of w.
- Common to standardize a r.v. by subtracting its expectation and dividing by its standard deviation

$$Z = \frac{X - E[X]}{\sqrt{Var[X]}}$$

The Central Limit Theorem

Theorem

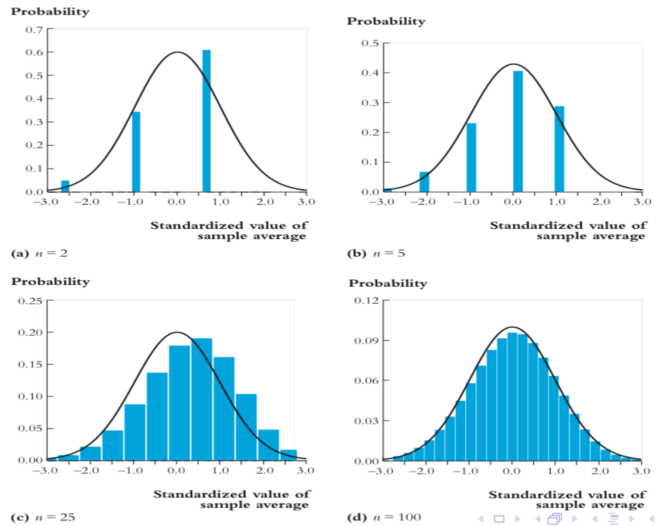
Let $X_1, ..., X_n$ be an i.i.d draws from a distribution with sample size nwith mean μ and $0 < \sigma^2 < \infty$, then

$$\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

- Because we don't have to make specific assumption about the distribution of X_i, so whatever the distribution of X_i, when n is big,
  - the standardized $\overline{X}_n \sim N(0,1)$
  - $\overline{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

The Central Limit Theorem

FIGURE 2.9 Distribution of the Standardized Sample Average of n Bernoulli Random Variables with $p = 0.78$

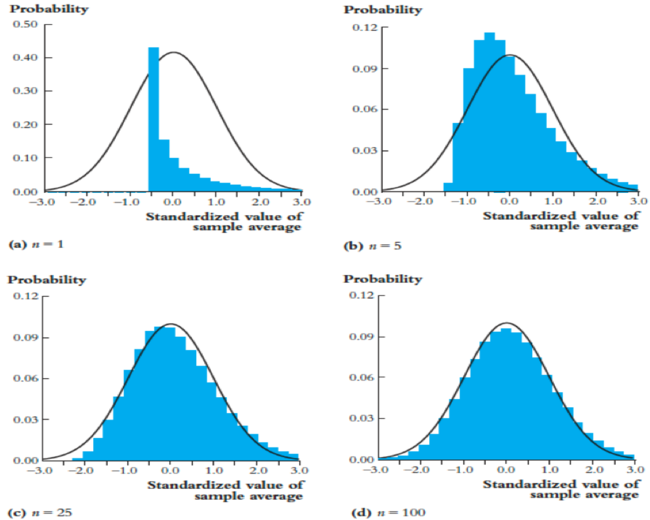


How large is "large enough" ?

- How large is large enough ?
  - how large must n be for the distribution of $\overline{Y}$ to be approximately normal?
- The answer: it depends.
  - if Y_i are themselves normally distributed, then $\overline{Y}$ is exactly normally distributed for all n.
  - if Y_i themselves have a distribution that is far from normal, then this approximation can require $n = 30$ or even more.

How large is "large enough"?

FIGURE 2.10 Distribution of the Standardized Sample Average of n Draws from a Skewed Distribution



Statistical Inference: Estimation, Confident Intervals and Testing

Statistical Inference

- Inference
  - What is our best guess about some quantity of interest?
  - What are a set of plausible values of the quantity of interest?
- **Compare estimators, such as** in an experiment
  - we use simple difference in sample means?
  - or the post-stratification estimator, where we estimate the estimate the difference among two subsets of the data (male and female, for instance) and then take the weighted average of the two variable
  - which is better? how could we know?

Inference: from Samples to Population

- Our focus: $\{Y_1, Y_2, ..., Y_n\}$ are i.i.d. draws from $f(y)$ or $F(Y)$, thus population distribution.
- Statistical inference or learning is using samples to infer $f(y)$.
- two ways
  - Parametric
  - Non-parametric

Point estimation

- Point estimation: providing a single "best guess" as to the value of some fixed, unknown quantity of interest, θ, which is is a feature of the population distribution, $f(y)$.
- Examples
  - $\mu = E[Y]$
  - $\sigma^2 = Var[Y]$
  - $\mu_y - \mu_x = E[Y] - E[X]$

Estimator and Estimate

Definition

Given a random sample$\{Y_1, Y_2, ..., Y_n\}$ drawn from a population distribution that depends on an unknown parameter θ, and an ***estimator*** $\hat{\theta}$ is a function of the sample: thus $\hat{\theta}_n = h(Y_1, Y_2, ..., Y_n)$

- An estimator is a r.v. because it is a function of r.v.s.
  - $\{\hat{\theta}_1, \hat{\theta}_2, ..., \hat{\theta}_n\}$ is a sequence of r.v.s, so it has convergence in probability/distribution.
- Question: what is the difference between an estimator and an statistic?

Definition

An **estimate** is the numerical value of the estimator when it is actually computed using data from a specific sample. Thus if we have the actual data $\{y_1, y_2, ..., y_n\}$,then $\hat{\theta} = h(y_1, y_2, ..., y_n)$

Example

Three Characteristics of an Estimator

- let $\hat{}_Y$ denote some estimator of μ_Y and $E(\hat{\mu}_Y)$ is the mean of the sampling distribution of $\hat{\mu}_Y$,

1. **Unbiasedness:** the estimator of μ_Y is *unbiased* if

$$E(\hat{\mu}_Y) = \mu_Y$$

2. **Consistency:**the estimator of μ_Y is *consistent* if

$$\hat{\mu}_Y \xrightarrow{p} \mu_Y$$

3. **Efficiency:**Let $\tilde{\mu}_Y$ be another estimator of μ_Y and suppose that both $\tilde{\mu}_Y$ and $\hat{\mu}_Y$ are unbiased.Then $\hat{\mu}_Y$ is said to be more *efficient* than $\hat{\mu}_Y$

$$var(\hat{\mu}_Y) < var(\tilde{\mu}_Y)$$

   - Comparing variances is difficult if we do not restrict our attention to unbiased estimators because we could always use a trivial estimator with variance zero that is biased.

Properties of the sample mean

1. Let μ_Y and σ_Y^2 denote the mean and variance of Y_i, then

$$E(\overline{Y}) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \mu_Y$$

   so $\overline{Y}$ is an *unbiased* estimator of μ_Y.

2. Based on the L.L.N., $\overline{Y} \xrightarrow{p} \mu_Y$, so $\overline{Y}$ is also *consistent*.
3. the variance of sample mean

$$Var(\overline{Y}) = var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i) = \frac{\sigma_Y^2}{n}$$

4. the standard deviation of the sample mean is $\sigma_{\overline{Y}} = \frac{\sigma_Y}{\sqrt{n}}$

Properties of the sample mean

- Because efficiency entails a comparison of estimators, we need to specify the estimator or estimators to which Y is to be compared.
  - Let $\widetilde{Y} = \frac{1}{n}\left(\frac{1}{2}Y_1 + \frac{3}{2}Y_2 + \frac{1}{2}Y_3 + \frac{3}{2}Y_4 + ... + \frac{1}{2}Y_{n-1} + \frac{3}{2}Y_n\right)$
    - $Var(\widetilde{Y}) = 1.25\frac{\sigma_Y^2}{n} > \frac{\sigma_Y^2}{n} = Var(\overline{Y})$
    - Thus $\overline{Y}$ is more efficient than $\widetilde{Y}$

Properties of the Sample Variance

- Let μ_Y and σ_Y^2 denote the mean and variance of Y_i , then the sample variance: $S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$

1. $E(S_Y^2) = \sigma_Y^2$, thus S^2 is an *unbiased* estimator of σ_Y^2. It is also the reason why the average uses the divisor $n-1$ instead of n.
2. $S_Y^2 \xrightarrow{P} \sigma_Y^2$, thus the sample variance is a consistent estimator of the population variance.
    - Because $\sigma_{\overline{Y}} = \frac{\sigma_Y}{\sqrt{n}}$, so the statement above justifies using $\frac{S_Y}{\sqrt{n}}$ as an estimator of the standard deviation of the sample mean, $\sigma_{\overline{Y}}$.
    - It is called **the standard error** of the sample mean and it dented $SE[\overline{Y}]$ or $\hat{\sigma}_{\overline{Y}}$.

Interval Estimation and Confidence Intervals

Interval Estimation

- A point estimate provides no information about how close the estimate is "likely" to be to the population parameter.
- We cannot know how close an estimate for a particular sample is to the population parameter because the population is unknown.
- A different (complementary) approach to estimation is to produce ***a range of values*** that will contain the truth with some fixed probability.

What is a Confidence Interval?

Definition

A $100(1-\alpha)\%$ confidence interval for a population parameter θ is an interval $C_n = (a, b)$, where $a = a(Y_1, ..., Y_n)$ and $b = b(Y_1, ..., Y_n)$ are functions of the data such that

$$P(a < \theta < b) = 1 - \alpha$$

- In general, this confidence level is $1-\alpha$; where α is called **significance level**.

Interval Estimation and Condence Intervals

- Suppose the population has a normal distribution $N(\mu, \sigma^2)$ and let $Y_1, Y_2, ..., Y_n$ be a random sample from the population.
  - Then the sample mean has a normal distribution: $\overline{Y} \sim N(\mu, \frac{\sigma^2}{n})$
  - The standardized sample mean $\overline{Z}$ is given by: $\overline{Z} = \frac{\overline{Y}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$
- Then $\theta = \overline{Z}$, then $P(a < \theta < b) = 1 - \alpha$ turns into

$$a < \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} < b$$

then it follows that

$$P(\overline{Y} - a\sigma/\sqrt{n} < \mu < \overline{Y} + b\sigma/\sqrt{n}) = 1 - \alpha$$

- The random interval contains the population mean with a probability $1 - \alpha$.

Interval Estimation and Condence Intervals

- Two cases: σ is known and unknown
- When σ is known, for example,$\sigma = 1$, thus $Y \sim N(\mu, 1)$,
- then $\overline{Y} \sim N(\mu, \frac{\sigma^2}{n} = \frac{1}{n})$
- From this, we can standardize $\overline{Y}$, and, because the standardized version of $\overline{Y}$ has a standard normal distribution, and we let $\alpha = 0.05$, then we have

$$P(-1.96 < \frac{\overline{Y} - \mu}{1/\sqrt{n}} < 1.96) = 1 - 0.05$$

- The event in parentheses is identical to the event $\overline{Y} - 1.96/\sqrt{n} \leq \mu \leq \overline{Y} + 1.96/\sqrt{n}$, so

$$P(\overline{Y} - 1.96/\sqrt{n} \leq \mu \leq \overline{Y} + 1.96/\sqrt{n}) = 0.95$$

- The interval estimate of μ may be written as $[\overline{Y} - 1.96/\sqrt{n}, \overline{Y} + 1.96/\sqrt{n}]$

Interval Estimation and Condence Intervals

- When σ is unknown, we must use an estimate S , denote the sample standard deviation, replacing unknown σ

$$P(\overline{Y} - 1.96 S/\sqrt{n} \leq \mu \leq \overline{Y} + 1.96 S/\sqrt{n}) = 0.95$$

- This could not work because S is not a constant but a r.v.

Interval Estimation and Condence Intervals

Definition

The **t-statistic** or **t-ratio**:

$$\frac{\overline{Y} - \mu}{SE(\overline{Y})} \sim t_{n-1}$$

- To construct a 95% confidence interval, let c denote the 97.5^{th} percentile in the t_{n-1} distribution.

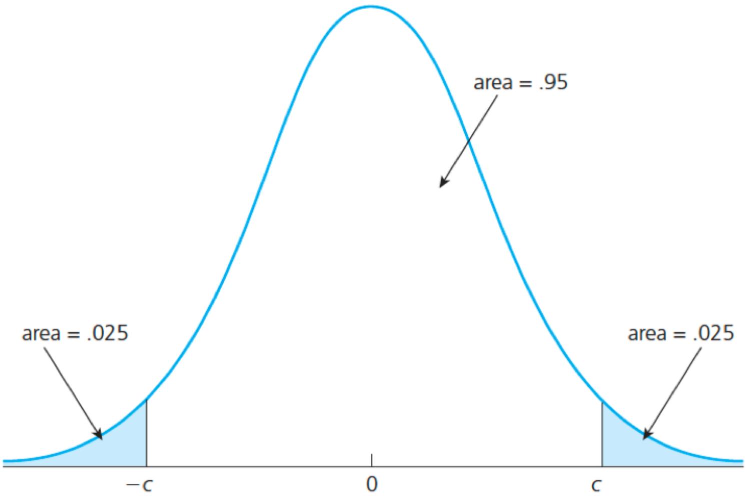
$$P(-c < t \leq c) = 0.95$$

  where $c_{\alpha/2}$ is the critical value of the t distribution.

- The condence interval may be written as $[\overline{Y} \pm c_{\alpha/2} S/\sqrt{n}]$

Interval Estimation and Condence Intervals

FIGURE C.4 The 97.5th percentile, c, in a t distribution.



A simple rule of thumb for a 95% confidence interval

- Caution! An often recited, but incorrect interpretation of a confidence interval is the following:
  - "I calculated a 95% confidence interval of [0.05,0.13], which means that there is a 95% chance that the true means is in that interval."
  - This is WRONG. actually μ either is or is not in the interval.
- The probabilistic interpretation comes from the fact that for 95% of all random samples, the constructed confidence interval will contain μ.

Interpreting the confidence interval

- Caution! An often recited, but incorrect interpretation of a confidence interval is the following:
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Hypothesis Testing

Hypothesis Testing

Definition

A hypothesis is a statement about a population parameter, thus θ. Formally, we want to test whether is significantly different from a certain value μ_0

$$H_0 : \theta = \mu_0$$

which is called **null hypothesis**. **The alternative hypothesis** is

$$H_1 : \theta \neq \mu_0$$

- If the value μ_0 does not lie within the calculated condence interval, then we **reject** the null hypothesis.
- If the value μ_0 lie within the calculated condence interval, then we **fail to reject** the null hypothesis.

General framework

- A hypothesis test chooses whether or not to reject the null hypothesis based on the data we observe.
- Rejection based on a test statistic

$$T_n = T(Y_1, ..., Y_n)$$

- The null/reference distribution is the distribution of T under the null.
- We'll write its probabilities as $P_0(T_n \leq t)$

Two Type Errors

- In both cases, there is a certain risk that our conclusion is wrong

Type I Error

A Type I error is when we reject the null hypothesis when it is in fact true.(“left-wing”)

- We say that the Lady is discerning when she is just guessing(null hypo: she is just guessing)

Type II Error

A Type II error is when we fail to reject the null hypothesis when it is false.(“right-wing”)

General framework

- A hypothesis test chooses whether or not to reject the null hypothesis based on the data we observe.
- Rejection based on a test statistic

$$T_n = T(Y_1, ..., Y_n)$$

- The null/reference distribution is the distribution of T under the null.
- We' ll write its probabilities as $P_0(T_n \leq t)$

P-Value

- To provide additional information, we could ask the question: What is the largest significance level at which we could carry out the test and still fail to reject the null hypothesis?
- We can consider the **p-value** of a test
  1. Calculate the t-statistic t
  2. The largest significance level at which we would fail to reject H_0 is the significance level associated with using t as our critical value

$$p - value = 1 - \Phi(t)$$

  where denotes the standard normal c.d.f.(we assume that n is large enough)

P-Value

- Suppose that $t = 1.52$, then we can find the largest significance level at which we would fail to reject H_0

$$p - value = P(T > 1.52 \mid H_0) = 1 - \Phi(1.52) = 0.065$$

../../../../2017Fall/Econometrics/LeCSlides/Lec2/fig6.png

Comparing Means from Different Populations

An Example: Comparing Means from Different Populations

- Do recent male and female college graduates earn the same amount on average? This question involves comparing the means of two different population distributions.
- In an RCT, we would like to estimate the average causal effects over the population

$$ATE = ATT = E\{Y_i(1) - Y_i(0)\}$$

- We only have random samples and random assignment to treatment, then what we can estimate instead

$$\textit{difference in mean} = \overline{Y}_{treated} - \overline{Y}_{control}$$

- Under randomization, *difference-in-means* is a good estimate for the ATE.

Hypothesis Tests for the Difference Between Two Means

- To illustrate a test for the difference between two means, let mw be the mean hourly earning in the population of women recently graduated from college and let mm be the population mean for recently graduated men.
- Then the **null hypothesis** and **the two-sided alternative hypothesis** are

$$\begin{aligned} H_0 : \mu_m &= \mu_w \\ H_1 : \mu_m &\neq \mu_w \end{aligned}$$

- Consider the null hypothesis that mean earnings for these two populations differ by a certain amount, say d_0. The null hypothesis that men and women in these populations have the same mean earnings corresponds to H_0 : $H_0 : d_0 = \mu_m - \mu_w = 0$

The Difference Between Two Means

- Suppose we have samples of n_m men and n_w women drawn at random from their populations. Let the sample average annual earnings be $\overline{Y}_m$ for men and $\overline{Y}_w$ for women. Then an estimator of $\mu_m - \mu_w$ is $\overline{Y}_m - \overline{Y}_w$.
- Let us discuss the distribution of $\overline{Y}_m - \overline{Y}_w$.

$$\sim N(\mu_m - \mu_w, \frac{\sigma_m^2}{n_m} + \frac{\sigma_w^2}{n_w})$$

- if $\sigma_m^2$and σ_w^2 are known, then the this approximate normal distribution can be used to compute p-values for the test of the null hypothesis. In practice, however, these population variances are typically unknown so they must be estimated.
- Thus the *standard error* of $\overline{Y}_m - \overline{Y}_w$ is

$$SE(\overline{Y}_m - \overline{Y}_w) = \sqrt{\frac{s_m^2}{n_m} + \frac{s_w^2}{n_w}}$$

The Difference Between Two Means

- The t-statistic for testing the null hypothesis is constructed analogously to the t-statistic for testing a hypothesis about a single population mean, thus *t-statistic* for comparing two means is

$$t = \frac{\overline{Y}_m - \overline{Y}_w - d_0}{SE(\overline{Y}_m - \overline{Y}_w)}$$

- If both $n_m$and n_m are large, then this t-statistic has a standard normal distribution when the null hypothesis is true.

Confidence Intervals for the Difference Between Two Population Means

- the 95% two-sided confidence interval for d consists of those values of d within ± 1.96 standard errors of $\overline{Y}_m - \overline{Y}_w$, thus $d = \mu_m - \mu_w$ is

$$(\overline{Y}_m - \overline{Y}_w) \pm 1.96 SE(\overline{Y}_m - \overline{Y}_w)$$

Wrap Up