Lecture 4: Hypothesis Tests in OLS Regression

Introduction to Econometrics,Fall 2023

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Review the last lecture

Omitted Variable Bias and M-OLS

- **Omitted Variable Bias(OVB)** violates the first Least Squares Assumption:

$$E(u_i|X_i) = 0$$

  - It will make Simple OLS estimation **biased** and **inconsistent**.
  - If the omitted variable can be observed and measured, then we can put it into the regression, thus **control** it to eliminate the bias.
- We have to extend **the Simple OLS regression** to **the Multiple one**.

Multiple regression model with k regressors

- The multiple regression model is

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + u_i, i = 1, \ldots, n$$

- where
  - Y_i is the *dependent variable*
  - $X_1, X_2, \ldots X_k$ are the *independent variables(includes one treatment variable and some control variables)*
  - $\beta_i, j = 1 \ldots k$ are slope coefficients on X_i corresponding.
  - β_0 is the estimate *intercept*, the value of Y when all $X_j = 0, j = 1 \ldots k$
  - u_i is the regression error term.

Multiple Regression: Assumptions

If the four least squares assumptions in the multiple regression model hold:

- Assumption 1: The conditional distribution of u_i given $X_{1i}, \ldots, X_{ki}$ has mean zero,thus

$$E[u_i | X_{1i}, \ldots, X_{ki}] = 0$$

- Assumption 2: $(Y_i, X_{1i}, \ldots, X_{ki})$ are i.i.d.
- Assumption 3: Large outliers are unlikely.
- Assumption 4: **No perfect multicollinearity**.

Then

- The OLS estimators $\hat{\beta}_0, \hat{\beta}_1 \ldots \hat{\beta}_k$ are *unbiased*.
- The OLS estimators $\hat{\beta}_0, \hat{\beta}_1 \ldots \hat{\beta}_k$ are *consistent*.
- The OLS estimators $\hat{\beta}_0, \hat{\beta}_1 \ldots \hat{\beta}_k$ are *normally distributed* in large samples.

Hypothesis Testing

Introduction: Class size and Test Score

Recall our simple OLS regression mode is

$$TestScore_i = \beta_0 + \beta_1 STR_i + u_i \tag{4.3}$$

Class Size and Test Score

Then we got the result of a simple OLS regression

$$\widehat{TestScore} = 698.9 - 2.28 \times STR, \;\; R^2 = 0.051, SER = 18.6$$

- **Don't forget**: the result are not obtained from the population **but from the sample**.
- How can you be sure about the result? In other words, *how confident* you can believe the result from the sample inferring to the population?
- If someone believes that cutting the class size will not help boost test scores. Can you reject the claim based your *scientific evidence-based* data analysis?
- This is the work of **Hypothesis Testing** in OLS regressions.

Review: Hypothesis Testing

- A hypothesis is (usually) an **assertion** or **statement** about **unknown population parameters** like θ.
- Suppose we want to test whether it is significantly different from a certain value μ_0
- Then **null hypothesis** is

$$H_0 : \theta = \mu_0$$

- The alternative hypothesis(two-sided) is

$$H_1 : \theta \neq \mu_0$$

- If the value μ_0 does not lie within the calculated confidence interval, then we *reject the null hypothesis.*
- If the value μ_0 lie within the calculated confidence interval, then we *fail to reject the null hypothesis.*

Review: Hypothesis Testing

- Most countries follow the rule of criminal trials: innocent until proven guilty(**疑罪从无**)
  - The jury or judge starts with the “null hypothesis” that the accused person is innocent.
  - The prosecutor wants to prove their hypothesis that the accused person is guilty.
  - In other words, they have to show strong evidence to make the jury or judge reject the “null hypothesis”.
- Likewise, our rule in econometrics is presumption of insignificance until proven.
  - At first researchers have to assume that there is **zero** impact of independent variable on dependent variable.
  - In order to prove the relationship between the independent variable and dependent variable, we must provide strong enough evidence to convince readers or policy makers to “reject” the assumption of a **zero** effect.

Review: Two Type Errors(两种错误)

- In both cases, there is a certain risk that our conclusion is wrong

	H_0 is true	H_A is true
Fail to reject H_O	Correct	Type II error
Reject H_O	Type I error	Correct

- Type I and Type II errors can not happen at the same time
- There is a trade-off between Type I and Type II errors

Review: Two Type Errors(两种错误)

- Question: Determine whether each situation belongs to Type I error or Type II error.
  - “宁可错杀一千，不能放过一个”
  - “宁可放过一千，不能错杀一个”

The Significance level(显著性水平)

- The significance level or size of a test, α, is the **maximum probability** of the Type I Error we tolerate.

$$P(\textit{Type I error}) = P(\textit{reject } H_0 \mid H_0 \textit{ is true}) = \alpha$$

- In social science, the usual significance level is set at 5%. A less rigorous standard is 10%, whereas a more stringent one is 1%.

The Power of the Test

- The power of a test, is $1 - \beta$, where β is the **maximum probability** of the Type II Error.

$$1 - P(\textit{Type II error}) = 1 - P(\textit{reject } H_0 \mid H_1 \textit{ is true}) = 1 - \beta$$

- In social science, the usual significance level is set at 5%. A less rigorous standard is 10%, whereas a more stringent one is 1%.

Review: Hypothesis Testing of Population Mean

- Recall: The **Student t** distribution can be obtained from a standard normal and a chi-square random variable.Let Z have a standard normal distribution, let X have a chi-square distribution with m degrees of freedom and assume that Z and X are independent. Then the random variable

$$T = \frac{Z}{\sqrt{X/n}}$$

  has has a t-distribution with m degrees of freedom, denoted as $T \sim t_n$
- The shape of the **t-distribution** is similar to that of a standard normal distribution, except that the t-distribution has more probability mass in the tails.

Review: Hypothesis Testing of Population Mean

- If the standard deviation of the population is unknown,then the

$$\frac{\overline{Y} - \mu_{Y,c}}{\sqrt{s_Y^2/n}} \rightarrow t_{n-1}$$

Review: Hypothesis Testing of Population Mean

- Let $\mu_{Y,c}$ is a specific value to which the population mean equals(thus we suppose)
  - **the null hypothesis**:
$$H_0 : E(Y) = \mu_{Y,c}$$
  - **the alternative hypothesis(two-sided)**:
$$H_1 : E(Y) \neq \mu_{Y,c}$$

Review: Hypothesis Testing of Population Mean

- Step 1 Compute the *sample mean* $\overline{Y}$
- Step 2 Compute the *standard error* of $\overline{Y}$, recall

$$SE(\overline{Y}) = \frac{s_Y}{\sqrt{n}}$$

- Step 3 Compute the *t-statistic* actually computed

$$t^{act} = \frac{\overline{Y}^{act} - \mu_{Y,c}}{SE(\overline{Y})}$$

- Step 4 Compute the p-value(optional)

$$\text{p-value} = 2\Phi(-|t^{act}|)$$

- Step 5 See if we can **Reject the null hypothesis** at a certain significance level α,like 5%, or p-value is less than significance level.

$$|t^{act}| > critical\ value \text{ or } p - value < significance\ level$$

Simple OLS: Hypotheses Testing

- A Simple OLS regression

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- This is the population regression equation and the key **unknown population parameters** is β_1.
- Then we would like to test whether β_1 equals to a specific value $\beta_{1,s}$ or not
    - **the null hypothesis**:

$$H_0 : \beta_1 = \beta_{1,s}$$

    - **the alternative hypothesis**:

$$H_1 : \beta_1 \neq \beta_{1,s}$$

A Simple OLS: Hypotheses Testing

- Step1: Estimate $Y_i = \beta_0 + \beta_1 X_i + u_i$ by OLS to obtain $\hat{\beta}_1$
- Step2: Compute the *standard error* of $\hat{\beta}_1$
- Step3: Construct the *t-statistic*

$$t^{act} = \frac{\hat{\beta}_1 - \beta_{1,c}}{SE\left(\hat{\beta}_1\right)}$$

- Step4: *Reject the null hypothesis* if

$$|\ t^{act}\ | > critical\ value$$

$$or \quad p - value < significance\ level$$

Recall: General Form of the t-statistics

$$t = \frac{\textit{estimator} - \textit{hypothesized value}}{\textit{standard error of the estimator}}$$

- Now the key unknown statistic is the **standard error**(S.E).

The Standard Error of $\hat{\beta}_1$

- **Recall** from the **Simple OLS Regression**
  - if the least squares assumptions hold, then in large samples $\hat{\beta}_0$ and $\hat{\beta}_1$ have a joint normal sampling distribution,thus $\hat{\beta}_1$
  $$\hat{\beta}_1 \sim N(\beta_1, \sigma^2_{\hat{\beta}_1})$$
  - We also derived the form of the variance of the normal distribution, $\sigma^2_{\hat{\beta}_1}$ is
  $$\sigma_{\hat{\beta}_1} = \sqrt{\frac{1}{n}\frac{Var[(X_i - \mu_X)u_i]}{[Var(X_i)]^2}} \tag{4.21}$$
- The value of $\sigma_{\hat{\beta}_1}$ is **unknown** and can not be obtained *directly* by the data.
  - $Var[(X_i - \mu_X)u_i]$ and $[Var(X_i)]^2$ are both unknown.

The Standard Error of $\hat{\beta}_1$

- Because $Var(X) = EX^2 - (EX)^2$, then the *numerator* in the square root in (4.21) is

$$Var[(X_i - \mu_X)u_i] = E[(X_i - \mu_X)u_i]^2 - (E[(X_i - \mu_X)u_i])^2$$

- Based on the Law of Iterated Expectation(L.I.E), we have

$$E[(X_i - \mu_X)u_i] = E\left(E[(X_i - \mu_X)u_i]|X_i\right)$$

- Again by the 1st OLS assumption, thus $E(u_i|X_i) = 0$,

$$E[(X_i - \mu_X)u_i] = 0$$

- Then the second term in the equation above

$$Var[(X_i - \mu_X)u_i] = E[(X_i - \mu_X)u_i]^2$$

The Standard Error of $\hat{\beta}_1$

- Because $plim(\overline{X}) = \mu_X$, then we use $\overline{X}$ and $\hat{\mu}_i$ to replace μ_X and μ_i in (4.21)(in large sample), then

$$
\begin{aligned}
Var[(X_i - \mu_X)u_i] =& E[(X_i - \mu_X)u_i]^2 \\
=& E[(X_i - \mu_X)^2 u_i^2] \\
=& plim\left(\frac{1}{n-2}\sum_{i=1}^{n}(X_i - \overline{X})^2 \hat{u}^2\right)
\end{aligned}
$$

where $n - 2$ is the *freedom of degree*.

The Standard Error of $\hat{\beta}_1$

- Because $plim(s_x) = \sigma_x^2 = Var(X_i)$, then

$$
\begin{aligned}
Var(X_i) &= \sigma_x^2 \\
&= plim(s_x) \\
&= plim(\frac{n-1}{n}(s_x)) \\
&= \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2
\end{aligned}
$$

- Then the *denominator* in the square root in (4.21) is

$$
[Var(X_i)]^2 = plim[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2]^2
$$

The Standard Error of $\hat{\beta}_1$

- The **standard error** of $\hat{\beta}_1$ is an **estimator** of the standard deviation of the sampling distribution $\sigma_{\hat{\beta}_1}$, thus

$$SE\left(\hat{\beta}_1\right) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}} = \sqrt{\frac{1}{n} \times \frac{\frac{1}{n-2}\sum(X_i - \bar{X})^2 \hat{u}_i^2}{\left[\frac{1}{n}\sum(X_i - \bar{X})^2\right]^2}} \tag{5.4}$$

- Everything in the equation (5.4) are known now or can be obtained by calculation.
- Then we can construct a *t-statistic* and then make a hypothesis test

$$t = \frac{estimator - hypothesized\ value}{standard\ error\ of\ the\ estimator}$$

Application to Test Score and Class Size

```
. regress test_score class_size, robust

Linear regression                               Number of obs     =        420
                                                   F(1, 418)         =      19.26
                                                   Prob > F          =     0.0000
                                                   R-squared         =     0.0512
                                                   Root MSE          =     18.581
```

test_score	Coef.	Robust Std. Err.	t	P>|t|	[95% Conf.	Interval]
class_size	-2.279808	.5194892	-4.39	0.000	-3.300945	-1.258671
_cons	698.933	10.36436	67.44	0.000	678.5602	719.3057

- the OLS regression line

$$\widehat{TestScore} = \underset{(10.4)}{698.9} - \underset{(0.52)}{22.8} \times STR, \quad R^2 = 0.051, SER = 18.6$$

Testing a two-sided hypothesis concerning β_1

- **the null hypothesis** $H_0 : \beta_1 = 0$
  - It means that the class size will not affect the performance of students.
- **the alternative hypothesis** $H_1 : \beta_1 \neq 0$
  - It means that the class size do affect the performance of students (whatever positive or negative)
- Our primary goal is to **Reject the null**, and then say make a conclusion:
  - Class Size **does matter** for the performance of students.

Testing a two-sided hypothesis concerning β_1

- Step1: Estimate $\hat{\beta}_1 = -2.28$
- Step2: Compute the standard error: $SE(\hat{\beta}_1) = 0.52$
- Step3: Compute the *t-statistic*

$$t^{act} = \frac{\hat{\beta}_1 - \beta_{1,c}}{SE\left(\hat{\beta}_1\right)} = \frac{-2.28 - 0}{0.52} = -4.39$$

- Step4: Reject the null hypothesis if
  - $\mid t^{act} \mid = \mid -4.39 \mid > critical\ value = 1.96$
  - $p - value = 0 < significance\ level = 0.05$

Application to Test Score and Class Size

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. regress test_score class_size, robust

Linear regression                               Number of obs     =        420
                                                  F(1, 418)         =      19.26
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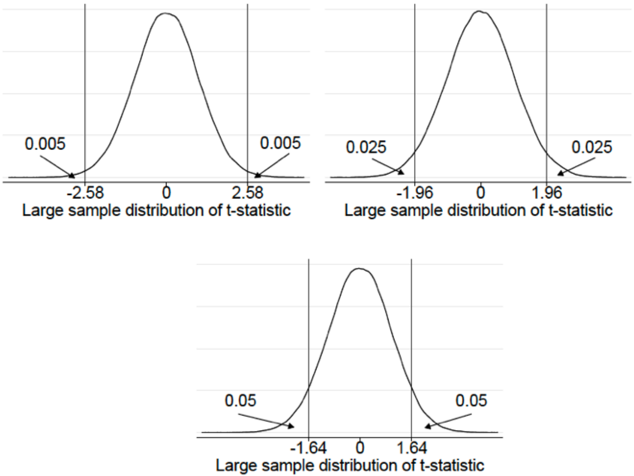
- We can reject the null hypothesis that $H_0 : \beta_1 = 0$, which means $\beta_1 \neq 0$ with a high probability(over 95%).
- It suggests that Class size **matters** the students' performance in a very high chance.

Critical Values of the t-statistic

The critical value of t-statistic depends on significance level α



1% and 10% significant levels

- Step4: Reject the null hypothesis at a **10%** significance level
  - $\mid t^{act} \mid = \mid -4.39 \mid > critical\ value = 1.64$
  - $p - value = 0.00 < significance\ level = 0.1$
- Step4: Reject the null hypothesis at a **1%** significance level
  - $\mid t^{act} \mid = \mid -4.39 \mid > critical\ value = 2.58$
  - $p - value = 0.00 < significance\ level = 0.01$

Two-Sided Hypotheses: β_1 in a certain value

- Step1: Estimate $\hat{\beta}_1 = -2.28$
- Step2: Compute the standard error: $SE(\hat{\beta}_1) = 0.52$
- Step3: Compute the t-statistic

$$t^{act} = \frac{\hat{\beta}_1 - \beta_{1,c}}{SE\left(\hat{\beta}_1\right)} = \frac{-2.28 - (-2)}{0.52} = -0.54$$

- Step4: **can't reject** the null hypothesis at 5% significant level because
  - $\mid t^{act} \mid = \mid -0.54 \mid < critical\ value = 1.96$
  - $p - value = 0.59 > significance\ level = 0.05$

Two-Sided Hypotheses : β_1 in a certain value

```
. lincom class_size-(-2)

 ( 1)  class_size = -2
```

test_score	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
(1)	-.2798083	.5194892	-0.54	0.590	-1.300945	.7413286

- We cannot reject the null hypothesis that $H_0 : \beta_1 = -2$.
- It suggests that *there is no enough evidence* to support the statement:
    - cutting class size in one unit will boost the test score in 2 points.

One-sided Hypotheses Concerning β_1

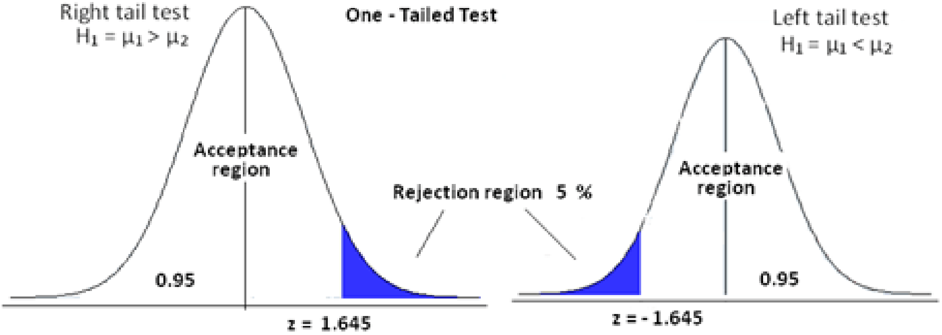
- Sometimes, we want to do a *one-sided Hypothesis testing*
- the null hypothesis is still unchanged $H_0 : \beta_1 = -2$
- **the alternative hypothesis** is $H_1 : \beta_1 < -2$
    - The statement is that reducing(or inversely increasing) class size will boost(or lower) student's performance.
    - More specifically,cutting class size in one unit will increase the test score in 2 points **at least**.
- Because the null hypothesis is the same for a one- and a two-sided hypothesis test, the construction of the t-statistic is the same.
- The difference between the two is the critical value and p-value.

One-sided Hypotheses Concerning β_1

- Step1: Estimate $\hat{\beta}_1 = -2.28$
- Step2: Compute the standard error: $SE(\hat{\beta}_1) = 0.52$
- Step3: Compute the t-statistic

$$t^{act} = \frac{\hat{\beta}_1 - \beta_{1,0}}{SE\left(\hat{\beta}_1\right)} = \frac{-2.28 - (-2)}{0.52} = -0.54$$

One-sided Hypotheses Concerning β_1



One-sided Hypotheses Concerning β_1

- Step4: under the circumstance, the critical value is not the -1.96 but -1.645 at 5% significant level.
- We can't reject the null hypothesis because

$$t^{act} = -0.54 > critical\ value = -1.645$$

- The p-value is not the $2\Phi(-|t^{act}|)$ now but $Pr(Z < t^{act}) = \Phi(t^{act})$.
- It suggests that *there is NO enough evidence* to support the statement:cutting class size in one unit will increase the test score in **2 points at least**.

One-sided Hypotheses Concerning β_1

- One-sided alternative hypotheses should be used only when there is a clear reason for doing so.
- This reason could come from economic theory, prior empirical evidence, or both.
- However, even if it initially seems that the relevant alternative is one-sided, upon reflection this might not necessarily be so.
- In practice, one-sided test is used much less than two-sided test.

Wrap up

- Hypothesis tests are useful if you have a specific null hypothesis in mind (as did our angry taxpayer).
- Being able to accept or reject this null hypothesis based on the statistical evidence provides a powerful tool for coping with the uncertainty inherent in using a sample to learn about the population.
- Yet, there are many times that no single hypothesis about a regression coefficient is dominant, and instead one would like to know a range of values of the coefficient that are consistent with the data.
- This calls for constructing a **confidence interval**.

Confidence Intervals

Introduction

- Because any statistical estimate of the slope β_1 necessarily has sampling uncertainty, we cannot determine the true value of β_1 exactly from a sample of data.
- It is possible, however, to use the OLS estimators and its standard error to construct a confidence interval for the slope β_1

CI for β_1

- Method for constructing a confidence interval for a population mean can be easily extended to constructing a confidence interval for a regression coefficient.
- Using a two-sided test, a hypothesized value for β_1 will be rejected at 5% significance level if

$$| t^{act} | > critical\ value = 1.96$$

- So $\hat{\beta}_1$ will be in the *confidence set* if $| t^{act} | \leq critical\ value = 1.96$
- Thus the 95% confidence interval for β_1 are within ± 1.96 standard errors of $\hat{\beta}_1$

$$\hat{\beta}_1 \pm 1.96 \cdot SE\left(\hat{\beta}_1\right)$$

CI for $\beta_{ClassSize}$

```
. regress test_score class_size, robust

Linear regression                               Number of obs     =        420
                                                F(1, 418)         =      19.26
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- Thus the 95% confidence interval for β_1 are within ± 1.96 standard errors of $\hat{\beta}_1$

$$\hat{\beta}_1 \pm 1.96 \cdot SE\left(\hat{\beta}_1\right) = -2.28 \pm (1.96 \times 0.519) = [-3.3, -1.26]$$

CI for predicted effets of changing X

- Consider changing X by a given amount, ΔX. The predicted change in Y associated with this change in X is $\beta_1 \Delta$.
- the 95% confidence interval for $\beta_1 \Delta X$ is

$$\hat{\beta}_1 \Delta X \pm 1.96 \cdot SE\left(\hat{\beta}_1\right) \times \Delta X$$

- eg reducing the student–teacher ratio by 2. then the 95% confidence interval is

$$[-3.3 \times 2, -1.34 \times 2] = [-6.6, -2.68]$$

Gauss-Markov theorem and Heteroskedasticity

Introduction

- Recall we discussed the properties of $\overline{Y}$ in Chapter 2.
    - an **unbiased** estimator of μ_Y
    - a **consistent** estimator of μ_Y
    - an **approximate normal sampling distribution** for large n

The Efficiency of $\overline{Y}$

- the fourth properties of $\overline{Y}$ in Chapter 3.
- the **Best Linear Unbiased Estimator(BLUE)**: $\overline{Y}$ is the most efficient estimator of μ_Y among all unbiased estimators that are weighted averages of $Y_1, ..., Y_n$, presented by $\hat{\mu}_Y = \frac{1}{n}\sum a_i Y_i$,thus,

$$Var(\overline{Y}) < Var(\hat{\mu}_Y)$$

Unnecessary Assumption for Simple OLS

- Three Simple OLS Regression Assumptions
  - Assumption 1
  - Assumption 2
  - Assumption 3
- Assumption 4: The error terms are **homoskedastic**

$$Var(u_i \mid X_i) = \sigma_u^2$$

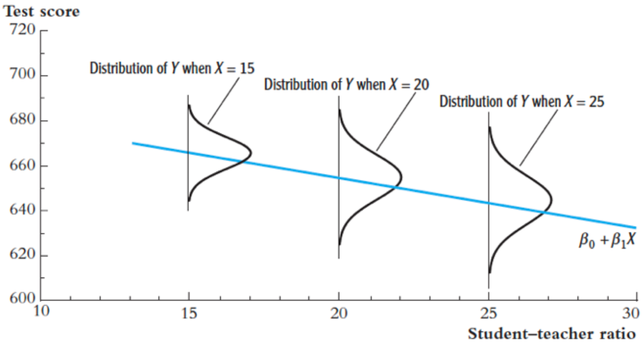
- Then $\hat{\beta}^{OLS}$ is the **Best Linear Unbiased Estimator(BLUE)**: it is the most efficient estimator of β_1 among all conditional unbiased estimators that are a linear function of $Y_1, Y_2, ..., Y_n$.

Heteroskedasticity & homoskedasticity

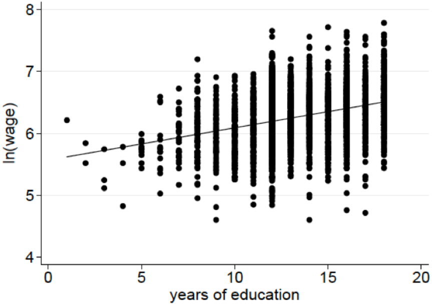
- The error term u_i is **homoskedastic** if the variance of the conditional distribution of u_i given X_i is constant for $i = 1, \ldots n$, in particular does not depend on X_i.
- Otherwise, the error term is **heteroskedastic**.

FIGURE 5.2 An Example of Heteroskedasticity

Like Figure 4.4, this shows the conditional distribution of test scores for three different class sizes. Unlike Figure 4.4, these distributions become more spread out (have a larger variance) for larger class sizes. Because the variance of the distribution of *u* given *X*, var(*u*|*X*), depends on *X*, *u* is heteroskedastic.



An Actual Example: the returns to schooling



- The spread of the dots around the line is clearly increasing with years of education X_i.
- Variation in (log) wages is higher at higher levels of education.
- This implies that

$$Var(u_i \mid X_i) \neq \sigma_u^2$$

Homoskedasticity: S.E.

- Recall the standard deviation of β_1, $\sigma^2_{\hat{\beta}_1}$, is

$$\sigma_{\hat{\beta}_1} = \sqrt{\frac{1}{n}\frac{Var[(X_i - \mu_X)u_i]}{[Var(X_i)]^2}} \tag{4.21}$$

- If u_i is **homoskedastic**, thus

$$Var(u_i|X_i) = \sigma^2_u Var(X_i) = \sigma^2_u$$

Homoskedasticity: S.E.

- The *numerator* in the square root in (4.21) can be transformed into

$$
\begin{aligned}
Var[(X_i - \mu_X)u_i] &= E[(X_i - \mu_X)u_i]^2 - (E[(X_i - \mu_X)u_i])^2 \\
&= E[(X_i - \mu_X)u_i]^2 \\
&= E[(X_i - \mu_X)^2 E(u_i^2|X_i)] \\
&= E[(X_i - \mu_X)^2 Var(u_i|X_i)] \\
&= \sigma_u^2 E[(X_i - \mu_X)^2]
\end{aligned}
$$

Homoskedasticity: S.E.

- Then the equation (4.21) turns into

$$\sigma_{\hat{\beta}_1} = \sqrt{\frac{1}{n}\frac{Var[(X_i - \mu_X)u_i]}{[Var(X_i)]^2}}$$
$$= \sqrt{\frac{1}{n}\frac{\sigma_u^2 Var(X_i)}{[Var(X_i)]^2}}$$
$$= \sqrt{\frac{1}{n}\frac{\sigma_u^2}{[Var(X_i)]}}$$

- So if we assume that the error terms are **homoskedastic**, then the **standard errors** of the OLS estimators β_1 simplify to

$$SE_{Homo}\left(\hat{\beta}_1\right) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}} = \sqrt{\frac{s^2_{\hat{u}}}{\sum(X_i - \bar{X})^2}}$$

Homoskedasticity: S.E.

- However,in many applications homoskedasticity is **NOT a plausible assumption**.
- If the error terms are *heteroskedastic*, then you use the *homoskedastic* assumption to compute the S.E. of $\hat{\beta}_1$. It will leads to
    - The standard errors are wrong (often too small)
    - The t-statistic does NOT have a $N(0, 1)$ distribution (also not in large samples).
    - But the estimating coefficients in OLS regression will not *change*.

Heteroskedasticity & homoskedasticity

- If the error terms are **heteroskedastic**, we should use the original equation of S.E.

$$SE_{Heter}\left(\hat{\beta}_1\right) = \sqrt{\hat{\sigma}^2_{\hat{\beta}_1}} = \sqrt{\frac{1}{n} \times \frac{\frac{1}{n-2}\sum(X_i - \bar{X})^2\hat{u}_i^2}{\left[\frac{1}{n}\sum(X_i - \bar{X})^2\right]^2}}$$

- It is called as *heteroskedasticity robust-standard errors*,also referred to as Eicker-Huber-White standard errors,simply **Robust-Standard Errors**
- In the case, it is not difficult to find that ***homoskedasticity* is just a special case of *heteroskedasticity***.

Heteroskedasticity & homoskedasticity

- Since homoskedasticity is a special case of heteroskedasticity, these heteroskedasticity robust formulas are also **valid** if *the error terms are homoskedastic*.
- Hypothesis tests and confidence intervals based on above SE's are *valid* both in case of homoskedasticity and heteroskedasticity.
- In reality, since in many applications homoskedasticity is not a plausible assumption, *it is best to use heteroskedasticity robust standard errors*. Using **robust standard errors** rather than **standard errors with homoskedasticity** will lead us lose nothing.

Heteroskedasticity & homoskedasticity

- It can be quite cumbersome to do this calculation by hand.Luckily,computer can help us do the job.
  - In `Stata`, the default option of regression is to assume homoskedasticity, to obtain heteroskedasticity robust standard errors use the option “robust”:

$$\textit{regress } y\ x\ ,\ \textit{robust}$$

  - In R, many ways can finish the job. A convenient function named `vcovHC()` is part of the package `sandwich`.

Test Scores and Class Size

```
. regress test_score class_size

      Source |       SS           df       MS      Number of obs   =       420
-------------+----------------------------------   F(1, 418)       =     22.58
       Model |  7794.11004         1  7794.11004   Prob > F        =    0.0000
    Residual |  144315.484       418  345.252353   R-squared       =    0.0512
-------------+----------------------------------   Adj R-squared   =    0.0490
       Total |  152109.594       419  363.030056   Root MSE        =    18.581

------------------------------------------------------------------------------
  test_score |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
  class_size |  -2.279808   .4798256    -4.75   0.000     -3.22298   -1.336637
       _cons |    698.933   9.467491    73.82   0.000     680.3231    717.5428
------------------------------------------------------------------------------

. regress test_score class_size, robust

Linear regression                               Number of obs     =        420
                                                F(1, 418)         =      19.26
                                                Prob > F          =     0.0000
                                                R-squared         =     0.0512
                                                Root MSE          =     18.581

------------------------------------------------------------------------------
             |               Robust
  test_score |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
  class_size |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
       _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

Test Scores and Class Size

```
. regress test_score class_size

      Source |       SS           df       MS      Number of obs   =       420
-------------+----------------------------------   F(1, 418)       =     22.58
       Model |  7794.11004         1  7794.11004   Prob > F        =    0.0000
    Residual |  144315.484       418  345.252353   R-squared       =    0.0512
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       Total |  152109.594       419  363.030056   Root MSE        =    18.581

------------------------------------------------------------------------------
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  class_size |  -2.279808   .4798256    -4.75   0.000     -3.22298   -1.336637
       _cons |    698.933   9.467491    73.82   0.000     680.3231    717.5428
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. regress test_score class_size, robust

Linear regression                               Number of obs     =        420
                                                F(1, 418)         =      19.26
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                                                Root MSE          =     18.581

------------------------------------------------------------------------------
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  test_score |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
  class_size |  -2.279808   .5194892    -4.39   0.000    -3.300945   -1.258671
       _cons |    698.933   10.36436    67.44   0.000     678.5602    719.3057
------------------------------------------------------------------------------
```

Wrap up: Heteroskedasticity in a Simple OLS

- If the error terms are heteroskedastic
  - The fourth simple OLS assumption is violated.
  - The Gauss-Markov conditions do not hold.
  - The OLS estimator is not BLUE (not most efficient).
- But (given that the other OLS assumptions hold)
  - The OLS estimators are still *unbiased*.
  - The OLS estimators are still *consistent*.
  - The OLS estimators are *normally distributed* in large samples

OLS with Multiple Regressors: Hypotheses tests

Recall: the Multiple OLS Regression

- The multiple regression model is

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \ldots + \beta_k X_{k,i} + u_i, i = 1, \ldots, n$$

- Four Basic Assumptions
  - Assumption 1 : $E[u_i \mid X_{1i}, X_{2i} \ldots, X_{ki}] = 0$
  - Assumption 2 : i.i.d sample
  - Assumption 3 : Large outliers are unlikely.
  - Assumption 4 : No perfect multicollinearity.
- The Sampling Distribution: the OLS estimators $\hat{\beta}_j$ for $j = 1, \ldots, k$ are approximately normally distributed in large samples.

Standard Errors for the Multiple OLS Estimators

- There is *nothing* conceptually different between the single- or multiple-regressor cases.
  - Standard Errors for a Simple OLS estimator β_1

$$SE\left(\hat{\beta}_1\right) = \hat{\sigma}_{\hat{\beta}_1}$$

  - Standard Errors for Mutiple OLS Regression estimators β_j

$$SE\left(\hat{\beta}_j\right) = \hat{\sigma}_{\hat{\beta}_j}$$

- Remind: since now the joint distribution is not only for (Y_i, X_i),but also for (X_{ij}, X_{ik}).
- The formula for the *standard errors* in Multiple OLS regression are related with a *matrix* named **Variance-Covariance matrix**

Hypothesis Tests for a Single Coefficient

- the *t-statistic* in Simple OLS Regression

$$t^{act} = \frac{\hat{\beta}_1 - \beta_{1,c}}{SE\left(\hat{\beta}_1\right)} \sim N(0,1)$$

- the *t-statistic* in Multiple OLS Regression

$$t = \frac{\hat{\beta}_j - \beta_{j,c}}{SE\left(\hat{\beta}_j\right)} \sim N(0,1)$$

Hypothesis testing for single coefficient

- $H_0 : \beta_j = \beta_{j,c}$ $H_1 : \beta_1 \neq \beta_{j,c}$
- Step1: Estimate $\hat{\beta}_j$, by run a multiple OLS regression

$$Y_i = \beta_0 + \beta_1 X_{1i} + ... + \beta_j X_{ji} + ... + \beta_k X_{ki} + u_i$$

- Step2: Compute the standard error of $\hat{\beta}_j$ (*requires matrix algebra*)
- Step3: Compute the t-statistic

$$t^{act} = \frac{\hat{\beta}_j - \beta_{j,c}}{SE\left(\hat{\beta}_j\right)}$$

- Step4: Reject the null hypothesis if
  - $\mid t^{act} \mid > critical\ value$
  - or if $p - value < significance\ level$

Confidence Intervals for a single coefficient

- Also the same as in a simple OLS Regression.
- $\hat{\beta}_j$ will be in the confidence set if $\mid t^{act} \mid \leq \textit{critical value} = 1.96$ at the 95% confidence level.
- Thus the 95% confidence interval for β_j are within ± 1.96 standard errors of $\hat{\beta}_j$

$$\hat{\beta}_j \pm 1.96 \cdot SE\left(\hat{\beta}_j\right)$$

Test Scores and Class Size

```
. regress test_score class_size el_pct,robust

Linear regression                               Number of obs     =        420
                                                F(2, 417)         =     223.82
                                                Prob > F          =     0.0000
                                                R-squared         =     0.4264
                                                Root MSE          =     14.464

------------------------------------------------------------------------------
             |               Robust
  test_score |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
  class_size |  -1.101296   .4328472    -2.54   0.011     -1.95213   -.2504616
      el_pct |  -.6497768   .0310318   -20.94   0.000     -.710775   -.5887786
       _cons |   686.0322   8.728224    78.60   0.000     668.8754    703.189
------------------------------------------------------------------------------
```

Case: Class Size and Test scores

- Does changing class size, while holding the percentage of English learners constant, have a statistically significant effect on test scores? (using a 5% significance level)
- $H_0 : \beta_{ClassSize} = 0 \ H_1 : \beta_{ClassSize} \neq 0$
- Step1: Estimate $\hat{\beta}_1 = -1.10$
- Step2: Compute the standard error: $SE(\hat{\beta}_1) = 0.43$
- Step3: Compute the t-statistic

$$t^{act} = \frac{\hat{\beta}_1 - \beta_{1,c}}{SE\left(\hat{\beta}_1\right)} = \frac{-1.10 - 0}{0.43} = -2.54$$

- Step4: Reject the null hypothesis if
    - $\mid t^{act} \mid = \mid -2.54 \mid > critical\ value.1.96$
    - $p - value = 0.011 < significance\ level = 0.05$

Tests of Joint Hypotheses: on 2 or more coefficients

- Can we just test one individual coefficient at a time?
- Suppose the angry taxpayer hypothesizes that neither the *student–teacher ratio* nor *expenditures per pupil* have an effect on test scores, once we control for the *percentage of English learners*.
- Therefore, we have to test a **joint null hypothesis** that both the coefficient on *student–teacher ratio* and the coefficient on *expenditures per pupil* are zero?

$$H_0 : \beta_{str} = 0 \;\&\; \beta_{expn} = 0,$$

$$H_1 : \beta_{str} \neq 0 \; and/or \; \beta_{expn} \neq 0$$

Testing 1 hypothesis on 2 or more coefficients

- If either t_{str} or t_{expn} exceeds 1.96, should we reject the null hypothesis?
- Assume that t_{str} and t_{expn} are *uncorrelated* at first:

$$
\begin{aligned}
&Pr(|t_{str}| > 1.96 \; and/or \; |t_{expn}| > 1.96) \\
&= 1 - Pr(|t_{str}| \leq 1.96 \; and \; |t_{expn}| \leq 1.96) \\
&= 1 - Pr(|t_{str}| \leq 1.96) * Pr\,|t_{expn}| \leq 1.96) \\
&= 1 - 0.95 \times 0.95 \\
&= 0.0975 > 0.05
\end{aligned}
$$

- We **cannot** reject the null hypothesis at 5% significant level now,even the single `t-test` for both variables can.

Testing 1 hypothesis on 2 or more coefficients

- If t_{str} and t_{expn} are correlated, then *it is more complicated* as simple t-statistic is not enough for hypothesis testing in Multiple OLS.
- In general, a joint hypothesis is a hypothesis that imposes two or more restrictions on the regression coefficients.

$$H_0 : \beta_j = \beta_{j,c}, \beta_k = \beta_{k,c}, ..., \textit{for a total of q restrictions}$$
$$H_1 : \textit{one or more of q restrictions under } H_0 \textit{ does not hold}$$

  - where $\beta_j, \beta_k, ...$ refer to different regression coefficients.
- When the regressors are highly correlated, we use **F-statistic**to testing joint hypotheses.

Unrestricted v.s Restricted model

- **The unrestricted model**: the model without any of the restrictions imposed. It contains all the variables.
- **The restricted model**: the model on which the restrictions have been imposed.
- And we want to test that $H_0 : \beta_1 = 0 \; and \; \beta_2 = 0$,then $H_1 : \beta_1 \neq 0 \; and/or \; \beta_2 \neq 0$ for the regression model

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \beta_3 X_{3,i} + u_i, i = 1, ..., n$$

- Then restricted model is

$$Y_i = \beta_0 + \beta_3 X_{3,i} + u_i$$

The F-statistic with q restrictions

- The F-statistic is computed using a simple formula based on the sum of squared residuals from two regressions.

$$F = \frac{\left(SSR_{\text{restricted}} - SSR_{\text{unrestricted}}\right)/q}{SSR_{\text{unrestricted}}/(n-k-1)}$$

- $SSR_{restricted}$ is the sum of squared residuals from the **restricted** regression.
- $SSR_{unrestricted}$ is the sum of squared residuals from the **full** model.
- q is the number of restrictions under the null.
- k is the number of regressors in the unrestricted regression.

The F-statistic and R^2

- An alternative equivalent formula for the_homoskedasticity-only F-statistic_ is based on the R2 of the two regressions:

$$F = \frac{\left(R^2_{\text{restricted}} - R^2_{\text{unrestricted}}\right)/q}{1 - R^2_{\text{unrestricted}}/\left(n - k - 1\right)}$$

- Only if the error terms are **homoskedastic**

$$Var(u_i \mid X_i) = \sigma_u^2$$

Testing 1 hypothesis on 2 or more coefficients

- Suppose we want to test

$$H_0 : \beta_1 = 0 \;\&\; \beta_2 = 0 \quad H_1 : \beta_1 \neq 0 \; and/or \; \beta_2 \neq 0$$

- Then the *F-statistic* can also combine the two *t-statistics*t_1 and t_2 as follows

$$F = \frac{1}{2}\left(\frac{t_1^2 + t_2^2 - 2\hat{\rho}_{t_1 t_2} t_1 t_2}{1 - \hat{\rho}_{t_1 t_2}^2}\right)$$

where $\hat{\rho}_{t_1 t_2}$ is an estimator of the correlation between the two t-statistics.

The heteroskedasticity-robust F-statistic with q restrictions.

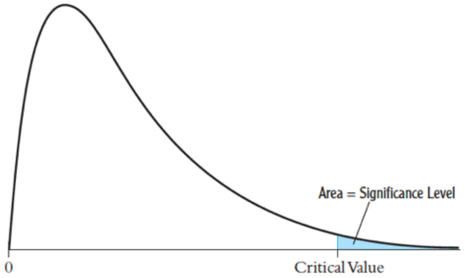
- Using matrix to show the form of the heteroskedasticity-robust F-statistic which is **beyond the scope of our class**.
- While,under the null hypothesis,regardless of whether the errors are homoskedastic or heteroskedastic, the F-statistic with q has a sampling distribution in large samples,

$$F - statistic \sim F_{q,\infty}$$

  - where q is the number of restrictions
- Then we can compute the F-statistic, the critical values from the table of the $F_{q,\infty}$ and obtain the p-value.

F-Distribution

TABLE 4 Critical Values for the $F_{m,\infty}$ Distribution



	Significance Level		
Degrees of Freedom	10%	5%	1%
1	2.71	3.84	6.63
2	2.30	3.00	4.61
3	2.08	2.60	3.78

Testing joint hypothesis with q restrictions

- $H_0 : \beta_j = \beta_{j,0}, ..., \beta_m = \beta_{m,0}$ for a total of q restrictions.
- H_1:at least one of q restrictions under H_0 does not hold.
- Step1: Estimate

$$Y_i = \beta_0 + \beta_1 X_{1i} + ... + \beta_j X_{ji} + ... + \beta_k X_{ki} + u_i$$

  by OLS
- Step2: Compute the **F-statistic**
- Step3 : Reject the null hypothesis if

$$F - Statistic > F^{act}_{q,\infty}$$

  or

$$p - value = Pr[F_{q,\infty} > F^{act}] <= significant\ level$$

Case: Class Size and Test Scores

- We want to test hypothesis that both the coefficient on *student–teacher ratio* and the coefficient on *expenditures per pupil* are zero?
    - $H_0 : \beta_{str} = 0 \ \& \beta_{expn} = 0$
    - $H_1 : \beta_{str} \neq 0 \ and/or \ \beta_{expn} \neq 0$
- The null hypothesis consists of two restrictions $q = 2$

Case: Class Size and Test Scores

```
. regress test_score class_size expn_stu el_pct,robust

Linear regression                               Number of obs     =        420
                                                F(3, 416)         =     147.20
                                                Prob > F          =     0.0000
                                                R-squared         =     0.4366
                                                Root MSE          =     14.353

------------------------------------------------------------------------------
             |               Robust
  test_score |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
  class_size |  -.2863992   .4820728    -0.59   0.553    -1.234002     .661203
    expn_stu |   .0038679   .0015807     2.45   0.015     .0007607    .0069751
      el_pct |  -.6560227   .0317844   -20.64   0.000    -.7185008   -.5935446
       _cons |   649.5779   15.45834    42.02   0.000     619.1917    679.9641
------------------------------------------------------------------------------

. test class_size expn_stu

 ( 1)  class_size = 0
 ( 2)  expn_stu = 0

       F(  2,   416) =    5.43
            Prob > F =    0.0047
```

- It can be shown that the F-statistic with two restrictions has an approximate $F_{2,\infty}$ distribution in large samples

$F = 5.43 > F_{2,\infty} = 4.61$ at 1% significant level

The "overall" regression F-statistic

- The "overall" F-statistic test the joint hypothesis that all the k slope coefficients are zero
  - $H_0 : \beta_j = \beta_{j,0}, ..., \beta_m = \beta_{m,0}$ for a total of $q = k$ restrictions.
  - H_1: at least one of $q = k$ restrictions under H_0 does not hold.

The "overall" regression F-statistic

```
. regress test_score class_size expn_stu el_pct,robust

Linear regression                               Number of obs     =        420
                                                F(3, 416)         =     147.20
                                                Prob > F          =     0.0000
                                                R-squared         =     0.4366
                                                Root MSE          =     14.353

------------------------------------------------------------------------------
             |               Robust
  test_score |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
  class_size |  -.2863992   .4820728    -0.59   0.553    -1.234002     .661203
    expn_stu |   .0038679   .0015807     2.45   0.015     .0007607    .0069751
      el_pct |  -.6560227   .0317844   -20.64   0.000    -.7185008   -.5935446
       _cons |   649.5779   15.45834    42.02   0.000     619.1917    679.9641
------------------------------------------------------------------------------

. test class_size expn_stu el_pct

 ( 1)  class_size = 0
 ( 2)  expn_stu = 0
 ( 3)  el_pct = 0

       F(  3,   416) =  147.20
            Prob > F =    0.0000
```

- The overall $F - Statistics = 147.2$ which indicates at least one coefficient can not be**ZERO**.

Case: Analysis of the Test Score Data Set

Introduction

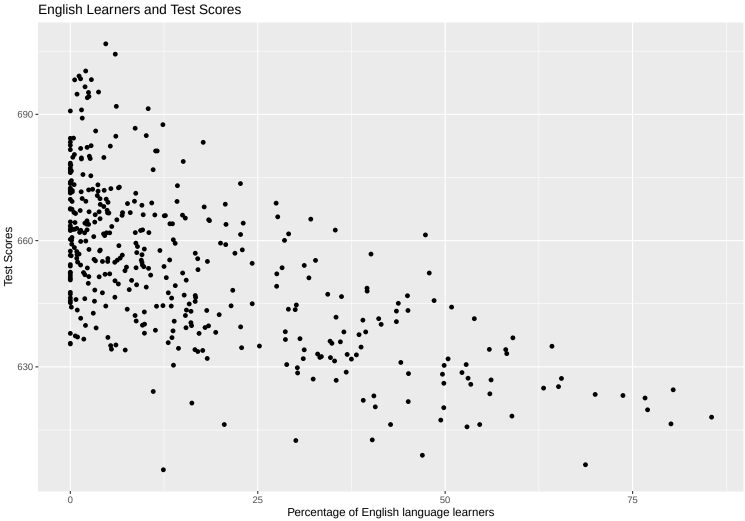
- How to use multiple regression in order to alleviate omitted variable bias and demonstrate how to report results.
- So far we have considered two variables that control for unobservable student characteristics which correlate with the student-teacher ratio *and* are assumed to have an impact on test scores:
  - *English*, the percentage of English learning students
  - *lunch*, the share of students that qualify for a subsidized or even a free lunch at school
  - *calworks*,the percentage of students that qualify for a income assistance program

Five different model equations:

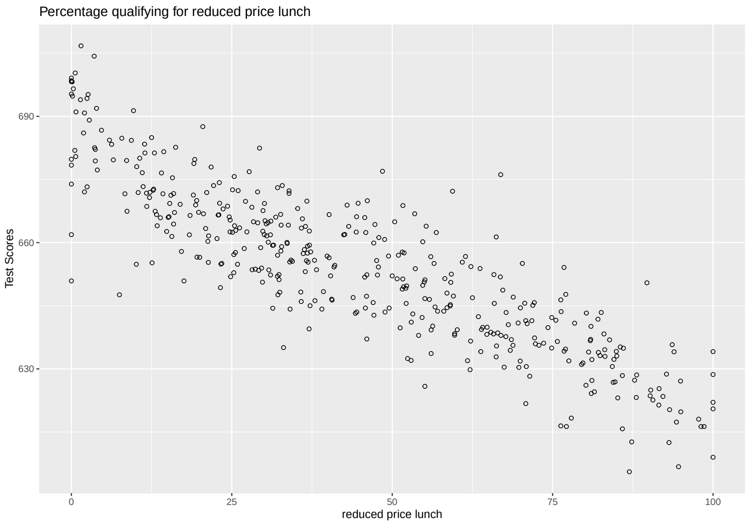
- We shall consider five different model equations:

$$
\begin{aligned}
(1) \quad & TestScore = \beta_0 + \beta_1 STR + u, \\
(2) \quad & TestScore = \beta_0 + \beta_1 STR + \beta_2 english + u, \\
(3) \quad & TestScore = \beta_0 + \beta_1 STR + \beta_2 english + \beta_3 lunch + u, \\
(4) \quad & TestScore = \beta_0 + \beta_1 STR + \beta_2 english + \beta_4 calworks + u, \\
(5) \quad & TestScore = \beta_0 + \beta_1 STR + \beta_2 english + \beta_3 lunch + \beta_4 calworks + u
\end{aligned}
$$

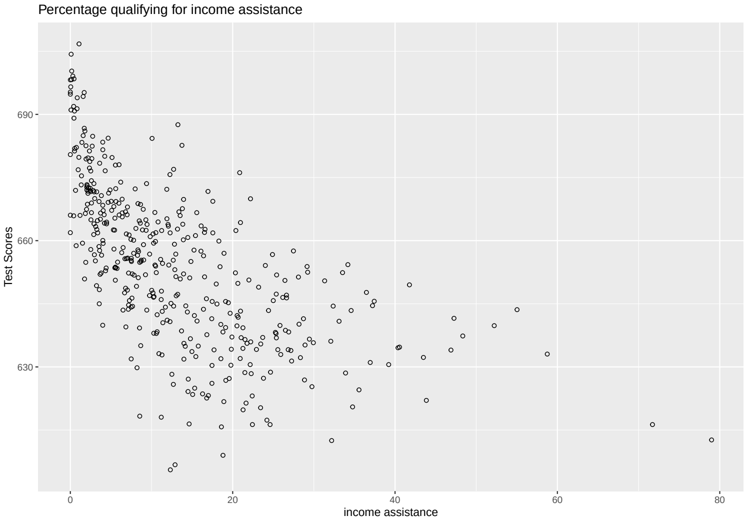
Scatter Plot: English learners and Test Scores



Scatter Plot: Free lunch and Test Scores



Scatter Plot: Income assistant and Test Scores



Correlations between Variables

- The correlation coefficients are following.

```
# estimate correlation between student characteristics and test scores
cor(CASchools$testscr, CASchools$el_pct)
```

```
## [1] -0.6441237
```

```
cor(CASchools$testscr, CASchools$meal_pct)
```

```
## [1] -0.868772
```

```
cor(CASchools$testscr, CASchools$calw_pct)
```

```
## [1] -0.6268534
```

```
cor(CASchools$meal_pct, CASchools$calw_pct)
```

Table 2

	Dependent Variable: Test Score	
	(1)	(2)
str	−2.280***	−1.101**
	(0.519)	(0.433)
el_pct		−0.650***
		(0.031)
Constant	698.933***	686.032***
	(10.364)	(8.728)
Observations	420	420
R^2	0.051	0.426
Adjusted R^2	0.049	0.424
F Statistic	22.575***	155.014***
Note:		*p<0.1; **p<0.05; ***p<0.01
		Robust S.E. are shown in the parentheses

Table 3

	Dependent Variable: Test Score			
	(1)	(2)	(3)	(4)
str	-2.280^{***}	-1.101^{**}	-0.998^{***}	-1.308^{***}
	(0.519)	(0.433)	(0.270)	(0.339)
el_pct		-0.650^{***}	-0.122^{***}	-0.488^{***}
		(0.031)	(0.033)	(0.030)
meal_pct			-0.547^{***}	
			(0.024)	
calw_pct				-0.790^{***}
				(0.068)
Constant	698.933^{***}	686.032^{***}	700.150^{***}	697.999^{***}
	(10.364)	(8.728)	(5.568)	(6.920)
Observations	420	420	420	420
R^2	0.051	0.426	0.775	0.629
Adjusted R^2	0.049	0.424	0.773	0.626

Table 4

	Dependent Variable: Test Score				
	(1)	(2)	(3)	(4)	(5)
str	-2.280^{***}	-1.101^{**}	-0.998^{***}	-1.308^{***}	-1.014^{***}
	(0.519)	(0.433)	(0.270)	(0.339)	(0.269)
el_pct		-0.650^{***}	-0.122^{***}	-0.488^{***}	-0.130^{***}
		(0.031)	(0.033)	(0.030)	(0.036)
meal_pct			-0.547^{***}		-0.529^{***}
			(0.024)		(0.038)
calw_pct				-0.790^{***}	-0.048
				(0.068)	(0.059)
Constant	698.933^{***}	686.032^{***}	700.150^{***}	697.999^{***}	700.392^{***}
	(10.364)	(8.728)	(5.568)	(6.920)	(5.537)
Observations	420	420	420	420	420
R^2	0.051	0.426	0.775	0.629	0.775
Adjusted R^2	0.049	0.424	0.773	0.626	0.773

The "Star War" and Regression Table

Dependent variable: average test score in the district.

Regressor	(1)	(2)	(3)	(4)	(5)
Student–teacher ratio (X_1)	−2.28** (0.52)	−1.10* (0.43)	−1.00** (0.27)	−1.31* (0.34)	−1.01* (0.27)
Percent English learners (X_2)		−0.650** (0.031)	−0.122** (0.033)	−0.488** (0.030)	−0.130** (0.036)
Percent eligible for subsidized lunch (X_3)			−0.547* (0.024)		−0.529* (0.038)
Percent on public income assistance (X_4)				−0.790** (0.068)	0.048 (0.059)
Intercept	698.9** (10.4)	686.0** (8.7)	700.2** (5.6)	698.0** (6.9)	700.4** (5.5)
Summary Statistics					
SER	18.58	14.46	9.08	11.65	9.08
$\bar{R}^2$	0.049	0.424	0.773	0.626	0.773
n	420	420	420	420	420

These regressions were estimated using the data on K–8 school districts in California, described in Appendix (4.1). Heteroskedasticity-robust standard errors are given in parentheses under coefficients. The individual coefficient is statistically significant at the *5% level or **1% significance level using a two-sided test.

Warp Up

- OLS is the most basic and important tool in econometricians' toolbox.
- The OLS estimators is unbiased, consistent and normal distributions under key assumptions.
- Using the hypothesis testing and confidence interval in OLS regression, we could make a more reliable judgment about the relationship between the treatment and the outcomes.

Appendix