Lecture 5: Nonlinear Regression Functions

Introduction to Econometrics,Spring 2025

Zhaopeng Qu

Nanjing University Business School

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1. Review of previous lecture
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Review of previous lecture

OLS Regression and Hypothsis Testing

- Since our β estimate comes from a sample, it contains **sampling error**. To determine the true relationship between treatment and outcomes, we must make statistical inferences from *our sample* to *the population*.
    - **Hypothesis Testing** is a statistical method that uses *sample data* to evaluate a hypothesis about a *population parameter*.
    - **Confidence Interval** is **a range of values** that is likely to contain the true population parameter.

OLS Regression and Hypothesis Testing

- **Hypothesis Testing** in OLS regressions
  - single coefficient: the **t-statistic**
  - potential assumption: large sample size $\Rightarrow$ **the normal distribution**
- The key component in obtaining the *t-statistic* is the **standard error**(S.E.), which is the estimation of **Standard Deviation** of estimated coefficients($\hat{\beta}$).
- **t-statistic** is calculated as:

$$t = \frac{\hat{\beta} - \beta}{SE(\hat{\beta})}$$

- **confidence interval** is calculated as:

$$\hat{\beta} \pm t_{critical} \times SE(\hat{\beta})$$

OLS Regression and Hypothsis Testing

- **Assumption 4**: When error terms are **homoskedastic** in OLS regression

$$Var(u_i \mid X_i) = \sigma_u^2$$

  the $\hat{\beta}^{OLS}$ is the **Best Linear Unbiased Estimator (BLUE)**.
- However, this assumption rarely holds in practice.
- Since **homoskedasticity** is merely a special case of **heteroskedasticity**, robust standard error formulas remain valid even when errors are homoskedastic.
- Therefore, we should generally use **heteroskedasticity-robust S.E.** in our analyses. Later, you'll learn additional methods for adjusting standard errors in other scenarios.

OLS Regression and Hypothsis Testing

- Two or more coefficients: the **F-statistic**
    - Testing individual coefficients with t-tests is insufficient for evaluating the joint significance of multiple coefficients.
    - The **F-statistic** follows an approximate χ^2 distribution, similar to other important test statistics such as the **Wald test**, **Likelihood ratio test**, and **Lagrange Multiplier(LM) test** (beyond the scope of this course).
- Through hypothesis testing and confidence intervals in OLS regression, we can make more reliable inferences about population-level relationships between treatments and outcomes.

Nonlinear Regression Functions

Introduction

- Recall the assumption of Linear Regression Model

Linear Regression Model

The observations, (Y_i, X_i) come from a random sample(i.i.d) and satisfy the linear regression equation,

$$Y_i = \beta_0 + \beta_1 X_{1,i} + ... + \beta_k X_{k,i} + u_i$$

- Everything what we have learned so far is under this assumption of linearity. But this linear approximation is not always a good one.

Introduction: Recall the whole picture

- A general formula for a population regression model may be

$$Y_i = f(X_{1,i}, X_{2,i}, ..., X_{k,i}) + u_i$$

- **Parametric methods**: assume that the function form(families) is known, we just need to assure(estimate) some unknown parameters in the function.
  - **Linear**(we just learned in the previous lectures)
  - **Nonlinear**(we will learn in this lecture)
- **Nonparametric methods**: assume that the function form is unknown or unnecessary to known.

Nonlinear Regression Functions

- How to extend a linear OLS model to be a nonlinear?

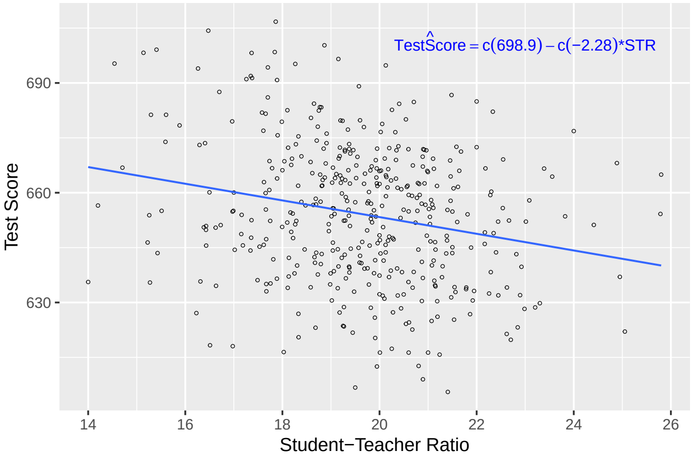
1. **Nonlinear in Xs**(the lecture now)
   - **Polynomials,Logarithms and Interactions**
   - The multiple regression framework can be extended to handle regression functions that are nonlinear in one or more X.
   - the difference from a standard multiple OLS regression is *how to explain estimating coefficients*.
2. **Nonlinear in** β or **Nonlinear in Y**(the next lecture)
   - **Discrete Dependent Variables** or **Limited Dependent Variables**.
   - Linear function in Xs is not a good prediciton function or Y.
   - Need a function which parameters enter nonlinearly, such as logisitic or negative exponential functions.
   - Then the parameters can not obtained by OLS estimation any more but *Nonlinear Least Squres* or *Maximum Likelyhood Estimation*.

Marginal Effect of X in Nonlinear Regression

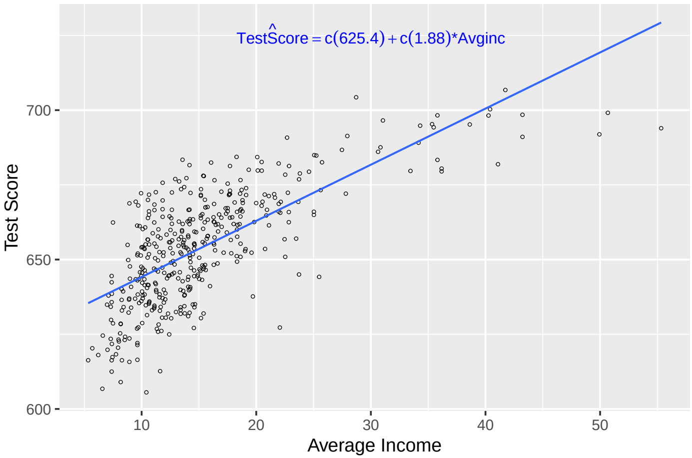
- If our regression model is linear: $Y_i = \beta_0 + \beta_1 X_{1,i} + ... + \beta_k X_{k,i} + u_i$
  - Then the **marginal effect** of X, thus *the effect of Y on a change in* X_j by 1 (unit) is **constant** and equals β_j:
$$\beta_j = \frac{\partial Y_i}{\partial X_{ji}}$$
- But if a relation between Y and X is **nonlinear**, thus $Y_i = f(X_{1,i}, X_{2,i}, ..., X_{k,i}) + u_i$
  - Then the marginal effect of X is not constant, but depends on the value of Xs(including X_i itself or/and other X_js) because
$$\frac{\partial Y_i}{\partial X_{ji}} = \frac{\partial f(X_{1,i}, X_{2,i}, ..., X_{k,i})}{\partial X_{ji}}$$
- The explaination of estimate coefficient β in nonlinear regression is **not as straightforward** as linear regression.

Nonlinear in Xs

The TestScore – STR relation looks linear (maybe)



But the TestScore – Income relation looks nonlinear



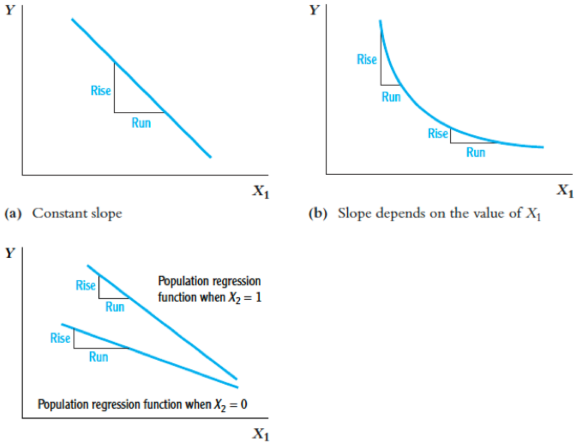
- **Overestimate** the true relationship when income is very high or very low and **underestimate** it for the middle income group.

Three Complementary Approaches:

1. **Polynomials in X**
   - The population regression function is approximated by a quadratic, cubic, or higher-degree polynomial.
2. **Logarithmic transformations**
   - Y and/or X is transformed by taking its logarithm
   - A **percentage** interpretation that makes sense in many applications
3. **Interactions**
   - The effect of X on Y depends on the value of another independent variable
   - often used in the analysis of hetergenous effects or channel effects.

Population Regression Functions with Different Slopes

FIGURE 8.1 Population Regression Functions with Different Slopes



(a) Constant slope

(b) Slope depends on the value of X_1

(c) Slope depends on the value of X_2

In Figure 8.1a, the population regression function has a constant slope. In Figure 8.1b, the slope of the population regression function depends on the value of X_1. In Figure 8.1c, the slope of the population regression function

The Effect of a Change in X in Nonlinear Functions

The Expected Change on Y of a Change in X_1 in the Nonlinear Regression Model (8.3)

KEY CONCEPT 8.1

The expected change in Y, ΔY, associated with the change in X_1, ΔX_1, holding $X_2, \dots, X_k$ constant, is the difference between the value of the population regression function before and after changing X_1, holding $X_2, \dots, X_k$ constant. That is, the expected change in Y is the difference:

$$\Delta Y = f(X_1 + \Delta X_1, X_2, \dots, X_k) - f(X_1, X_2, \dots, X_k). \quad (8.4)$$

The estimator of this unknown population difference is the difference between the predicted values for these two cases. Let $\hat{f}(X_1, X_2, \dots, X_k)$ be the predicted value of Y based on the estimator $\hat{f}$ of the population regression function. Then the predicted change in Y is

$$\Delta \hat{Y} = \hat{f}(X_1 + \Delta X_1, X_2, \dots, X_k) - \hat{f}(X_1, X_2, \dots, X_k). \quad (8.5)$$

Polynomials in X

Example: the TestScore-Income relation

- If a straight line is NOT an adequate description of the relationship between district income and test scores, what is?
- Two simple options
  - **Quadratic specification**:

$$TestScore_i = \beta_0 + \beta_1 Income_i + \beta_2 (Income_i)^2 + u_i$$

  - **Cubic specification**:

$$TestScore_i = \beta_0 + \beta_1 Income_i + \beta_2 (Income_i)^2 + \beta_3 (Income_i)^3 + u_i$$

- How to estimate these models?
  - We can see **quadratic** and **cubic** terms as two **independent variables** in the model.
  - Then the model turns into a **special form** of a multiple OLS regression model.

Estimation of the quadratic specification in R

```
#>
#> Call:
#>    felm(formula = testscr ~ avginc + I(avginc^2), data = ca)
#>
#> Residuals:
#>     Min       1Q   Median       3Q      Max
#> -44.416  -9.048   0.440   8.348  31.639
#>
#> Coefficients:
#>              Estimate Robust s.e t value Pr(>|t|)
#> (Intercept) 607.30174    2.90175 209.288   <2e-16 ***
#> avginc        3.85100    0.26809  14.364   <2e-16 ***
#> I(avginc^2)  -0.04231    0.00478  -8.851   <2e-16 ***
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> Residual standard error: 12.72 on 417 degrees of freedom
#> Multiple R-squared(full model): 0.5562   Adjusted R-squared: 0.554
#> Multiple R-squared(proj model): 0.5562   Adjusted R-squared: 0.554
#> F-statistic(full model, *iid*):261.3 on 2 and 417 DF, p-value: < 2.2e-16
```

Estimation of the cubic specification in R

```
#>
#> Call:
#>    felm(formula = testscr ~ avginc + I(avginc^2) + I(avginc3), data = ca)
#>
#> Residuals:
#>    Min      1Q  Median      3Q     Max
#> -44.28  -9.21   0.20    8.32   31.16
#>
#> Coefficients:
#>                  Estimate Robust s.e t value Pr(>|t|)
#> (Intercept)  6.001e+02  5.102e+00 117.615  < 2e-16 ***
#> avginc       5.019e+00  7.074e-01   7.095 5.61e-12 ***
#> I(avginc^2) -9.581e-02  2.895e-02  -3.309  0.00102 **
#> I(avginc3)   6.855e-04  3.471e-04   1.975  0.04892 *
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> Residual standard error: 12.71 on 416 degrees of freedom
#> Multiple R-squared(full model): 0.5584   Adjusted R-squared: 0.5552
#> Multiple R-squared(proj model): 0.5584   Adjusted R-squared: 0.5552
```

Table 1: Test Score and Income: Nonlinear OLS Regression

	Dependent Variable: Test Score		
	(1)	(2)	(3)
avginc	1.879***	3.851***	5.019***
	(0.113)	(0.267)	(0.704)
I(avginc^2)		−0.042***	−0.096***
		(0.005)	(0.029)
I(avginc^3)			0.001**
			(0.0003)
Constant	625.384***	607.302***	600.079***
	(1.863)	(2.891)	(5.078)
Observations	420	420	420
Adjusted R^2	0.506	0.554	0.555
F Statistic	430.830***	261.278***	175.352***
Note:	*p<0.1; **p<0.05; ***p<0.01		
	Robust S.E. are shown in the parentheses		

Quadratic vs Linear

- **Question**: Is the quadratic model better than the linear model?
- We can test the null hypothesis that the regression function is linear against the alternative hypothesis that it is quadratic:

$$H_0 : \beta_2 = 0 \; and \; H_1 : \beta_2 \neq 0$$

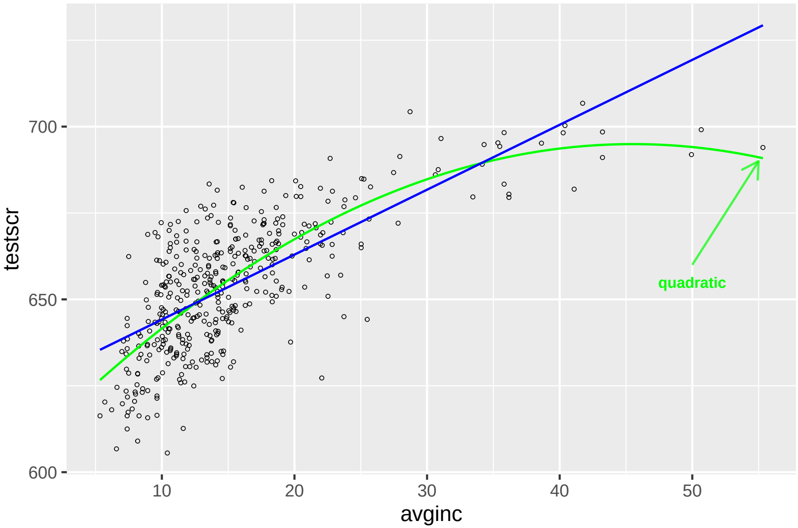
- the t-statistic

$$t = \frac{(\hat{\beta}_2 - 0)}{SE(\hat{\beta}_2)} = \frac{-0.0423}{0.0048} = -8.81$$

- Since $8.81 > 2.58$, we reject the null hypothesis (the linear model) at a 1% significance level.
- Based on the F-test, we can also reject the null hypothesis

$$F - statistic_{q=2,d=417} = 261.3, p - value \cong 0.00$$

Figure: Linear and Quadratic Regression



Interpreting the estimated quadratic regression function

- What is the **marginal effect** of X on Y in a quadratic regression function.
- The regression model now is

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + u_i$$

- The marginal effect of X on Y

$$\frac{\partial Y_i}{\partial X_i} = \beta_1 + 2\beta_2 X_i$$

- It means that the **marginal effect** of X on Y depends on the specific value of X_i

Interpreting the quadratic regression function

- The estimated regression function with a quadratic term of income is

$$\widehat{TestScore}_i = \underset{(2.90)}{607.3} + \underset{(0.27)}{3.85} \times income_i - \underset{(0.0048)}{0.0423} \times income_i^2.$$

- Suppose *the effect of an* $\$1000$ *increase on average income on test scores*
- A group: from $\$10,000$ per capita to $\$11,000$ per capita:

$$\begin{aligned}\Delta TestScore &= 607.3 + 3.85 \times 11 - 0.0423 \times (11)^2 \\ &- [607.3 + 3.85 \times 10 - 0.0423 \times (10)^2] \\ &= 2.96\end{aligned}$$

- B group: from $\$40,000$ per capita to $\$41,000$ per capita:

$$\begin{aligned}\Delta TestScore &= 607.3 + 3.85 \times 41 - 0.0423 \times (41)^2 \\ &- [607.3 + 3.85 \times 40 - 0.0423 \times (40)^2] \\ &= 0.42\end{aligned}$$

Quadratic vs Cubic

- **Question**: Is the cubic model better than the quadratic model?
- **Answer**: testing the null hypothesis that the regression function is *quadratic* against the alternative hypothesis that it is *cubic*:

$$H_0 : \beta_3 = 0 \; and \; H_1 : \beta_3 \neq 0$$

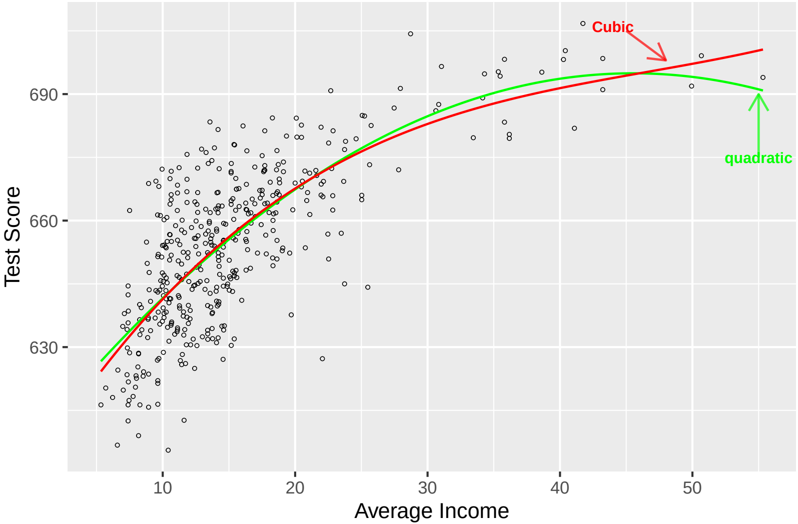
- the t-statistic

$$t = \frac{(\hat{\beta}_3 - 0)}{SE(\hat{\beta}_3)} = \frac{-0.001}{0.0003} = -3.33$$

- Since $3.33 > 2.58$, we reject the null hypothesis (the linear model) at a 1% significance level.
- the F-test also reject the null hypothesis: $\beta_1 = 0 = \beta_2 = \beta_3 = 0$:

$$F - statistic_{q=3,d=416} = 175.35, p - value \cong 0.00$$

Figure: Cubic and Quadratic Regression



Interpreting the estimated cubic regression function

- **The regression model now is**

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X_i^2 + \beta_3 X_i^3 + u_i$$

- **The marginal effect of X on Y**

$$\frac{\partial Y_i}{\partial X_i} = \beta_1 + 2\beta_2 X_i + 3\beta_3 X_i^2$$

Interpreting the estimated regression function

- **The estimated cubic model is**

$$\widehat{TestScore}_i = \underset{(5.83)}{600.1} + \underset{(0.86)}{5.02} \times income - \underset{(0.03)}{0.096} \times income^2 + \underset{(0.00047)}{0.00069} \times income^3.$$

- **A group**: from $10,000 per capita to $11,000 per capita:

$$\begin{aligned}\Delta TestScore &= 600.079 + 5.019 \times 11 - 0.096 \times (11)^2 + 0.001 \times (11)^3 \\ &\quad - [600.079 + 5.019 \times 10 - 0.096 \times (10)^2 + 0.001 \times (10)^3]\end{aligned}$$

- **B group**: from $40,000 per capita to $41,000 per capita:

$$\begin{aligned}\Delta TestScore &= 600.079 + 5.019 \times 41 - 0.096 \times (41)^2 + 0.001 \times (41)^3 \\ &\quad - [600.079 + 5.019 \times 40 - 0.096 \times (40)^2 + 0.001 \times (40)^3]\end{aligned}$$

Polynomials in X Regression Function

- Approximate the population regression function by a polynomial:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 X^2 ... + \beta_r X_i^r + u_i$$

- This is just the multiple linear regression model – except that the regressors are **powers of X**!
- Estimation, hypothesis testing, etc. proceeds as in the multiple regression model using OLS.
- Although, the coefficients are difficult to interpret, the regression function itself is interpretable.

Which degree polynomial should I use?

- How many powers of X should be included in a polynomial regression?
- The answer balances a **trade-off** between flexibility and statistical precision. (many ML or non-parametric or semi-parametric methods work on this)
    - Increasing the degree r introduces more flexibility into the regression function and allows it to match more shapes; a polynomial of degree r can have up to r - 1 bends (that is, inflection points) in its graph.
    - But increasing r means adding more regressors, which can reduce the precision of the estimated coefficients.

Which degree polynomial should I use?

- A practical way: asking whether the coefficients in the regression associated with the largest values of r are **zero**. If so, then these terms can be dropped from the regression.
- This procedure, which is called *sequential hypothesis testing*
  1. Pick a maximum value of r and estimate the polynomial regression for that r.
  2. Use the t-statistic to test whether the coefficient on X^r,β_r is **ZERO**.
  3. If reject, then the degree is r; if not then test the whether the coefficient on X^{r-1},β_{r-1} is **ZERO**.
  4. continue this procedure until the coefficient on the highest power in your polynomial is statistically significant.

Which degree polynomial should I use?

- The **initial degree** r of the polynomial is still missing.
- In many applications involving economic data, the nonlinear functions are smooth, that is, they do not have sharp jumps, or "spikes."
- If so, then it is appropriate to choose a small maximum degree for the polynomial, such as 2, 3, or 4.

Joint-Testing the population regression function

- If the population regression function is linear, then the higher-degree terms should not enter the population regression function.
- To perform hypothesis test

$$H_0 : \beta_2 = 0, \beta_3 = 0, ..., \beta_r = 0 \; and \; H_1 : at \; least \; one \; \beta_j \neq 0$$

- Because H_0 is a **joint null hypothesis** with $q = r - 1$ restrictions on the coefficients, it can be tested using the F-statistic.
- This can be easily extended to the case where the population regression function is quadratic or cubic.
- There are also several formal testing to determine the degree.
  - The Akaike Information Criterion(AIC)
  - The Bayes Information Criterion(BIC)

Wrap Up

- The nonlinear functions in Polynomials in Xs are just a special form of Multiple OLS Regression.
- If the true relationship between X and Y is nonlinear in polynomials in Xs, then a fully linear regression is misspecified - the functional form is wrong.
- The estimator of the effect on Y of X is biased and inconsistent which you can see it as a special case of OVB.
- Estimation, hypothesis testing, etc. proceeds as in the multiple regression model using OLS, which can also help us to tell the degrees of polynomial functions.
- The big difference is how to explained the estimate coefficients and make the predicted change in Y with a change in Xs.

Logarithms

Logarithmic functions of Y and/or X

- Another way to specify a nonlinear regression model is to use the natural logarithm of Y and/or X.
- $Ln(x)$ = the natural logarithm of x is the inverse function of the exponential function e^x, here $e = 2.71828$.

$$x = ln(e^x)$$

Review of the Basic Logarithmic functions

- **If X and a are variables, then we have**

$$
\begin{aligned}
ln(1/x) &= -ln(x) \\
ln(ax) &= ln(a) + ln(x) \\
ln(x/a) &= ln(x) - ln(a) \\
ln(x^a) &= aln(x)
\end{aligned}
$$

Logarithms and percentages

- Following the limitation rule of logarithms, we have

$$Ln(1+X) \cong X \; when \; X \; is \; very \; small$$

- Therefore,

$$ln(x+\Delta x) - ln(x) = ln\left(\frac{x+\Delta x}{x}\right)$$

Logarithms and percentages

- Following the limitation rule of logarithms, we have

$$Ln(1+X) \cong X \; when \; X \; is \; very \; small$$

- Therefore,

$$\begin{aligned} ln(x+\Delta x) - ln(x) &= ln\left(\frac{x+\Delta x}{x}\right) \\ &\cong \frac{\Delta x}{x} \; (when \; \frac{\Delta x}{x} \; is \; very \; small) \end{aligned}$$

- For example:

$$ln(1+0.01) = ln(101) - ln(100) = 0.00995 \cong 0.01$$

- Thus,logarithmic transforms permit modeling relations in **percentage** terms (like elasticities), rather than linearly.

The three log regression specifications:

Case	Population regression function
I.linear-log	

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Case	Population regression function
I.linear-log	$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$
II.log-linear	

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Case	Population regression function
I.linear-log	$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$
II.log-linear	$ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$
III.log-log	

The three log regression specifications:

Case	Population regression function
I.linear-log	$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$
II.log-linear	$ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$
III.log-log	$ln(Y_i) = \beta_0 + \beta_1 ln(X_i) + u_i$

- The interpretation of the slope coefficient **differs** in each case.
- The interpretation is found by applying the general "before and after" rule: "figure out the change in Y for a given change in X."(Key Concept 8.1 in S.W.pp301)

I. Linear-log population regression function

- **Regression Model:**

$$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$$

- **Change X ΔX:**

$$\Delta Y = [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)]$$

I. Linear-log population regression function

- **Regression Model:**

$$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$$

- **Change X ΔX:**

$$\begin{aligned} \Delta Y &= [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)] \\ &= \beta_1 [ln(X + \Delta X) - ln(X)] \end{aligned}$$

I. Linear-log population regression function

- **Regression Model:**

$$Y_i = \beta_0 + \beta_1 ln(X_i) + u_i$$

- **Change X ΔX:**

$$\begin{aligned} \Delta Y &= [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)] \\ &= \beta_1 [ln(X + \Delta X) - ln(X)] \\ &\cong \beta_1 \frac{\Delta X}{X} \end{aligned}$$

- **Note** $100 \times \frac{\Delta X}{X} = \textit{percentage change in } X$**, and** $\beta_1 \cong \frac{\Delta Y}{\frac{\Delta X}{X}}$
- **Interpretation of β_1: a 1 percent increase in X (multiplying X by 1.01 or $100 \times \frac{\Delta X}{X}$) is associated with a $0.01\beta_1$ or $\frac{\beta_1}{100}$ change in Y.**

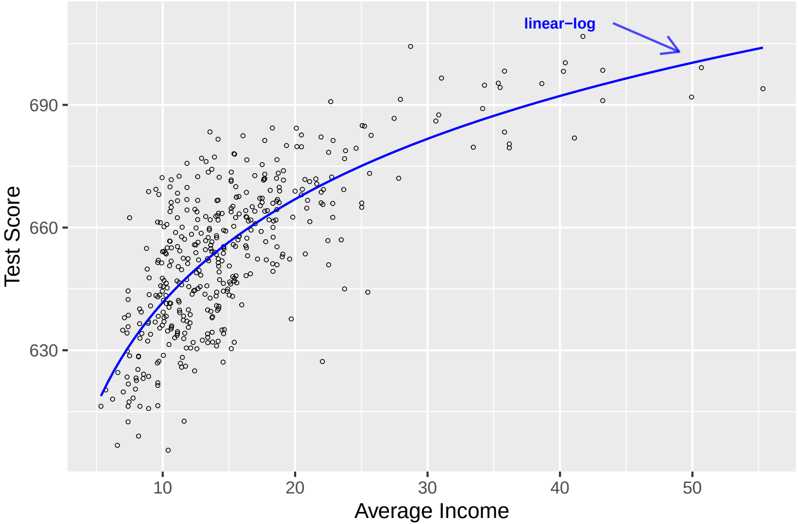
Example: the TestScore – log(Income) relation

- The OLS regression of ln(Income) on Testscore yields

$$\widehat{TestScore} = 557.8 + 36.42 \times ln(Income)$$
$$(3.8) \quad (1.4)$$

- **Interpretation** of β_1: a 1% increase in *Income* is associated with an increase in *TestScore* of 0.3642 **points**.
- **Calculate on the mean values**: Then if the mean of *Income* is \$25,000, then the predicted change in test score from a 1% increase in income is $\$25000 \times 1\% = \$250$ will increase *TestScore* by 0.3642 points.

Test scores: linear-log function



Case II. Log-linear population regression function

- **Regression model:**

$$ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

- **Change X:**

$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1(X + \Delta X)] - [\beta_0 + \beta_1 X]$$

Case II. Log-linear population regression function

- **Regression model:**

$$ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

- **Change X:**

$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1(X + \Delta X)] - [\beta_0 + \beta_1 X]$$
$$\Rightarrow ln(1 + \frac{\Delta Y}{Y}) = \beta_1 \Delta X$$

Case II. Log-linear population regression function

- **Regression model:**

$$ln(Y_i) = \beta_0 + \beta_1 X_i + u_i$$

- **Change X:**

$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1(X + \Delta X)] - [\beta_0 + \beta_1 X]$$
$$\Rightarrow ln(1 + \frac{\Delta Y}{Y}) = \beta_1 \Delta X$$
$$\Rightarrow \frac{\Delta Y}{Y} \cong \beta_1 \Delta X$$

- **So** $100\frac{\Delta Y}{Y} = percentage\ change\ in\ Y$ **and** $\beta_1 = \frac{\frac{\Delta Y}{Y}}{\Delta X}$
- **Then a change in X by one unit is associated with a** $\beta_1 \times 100$ **percent change in Y.**

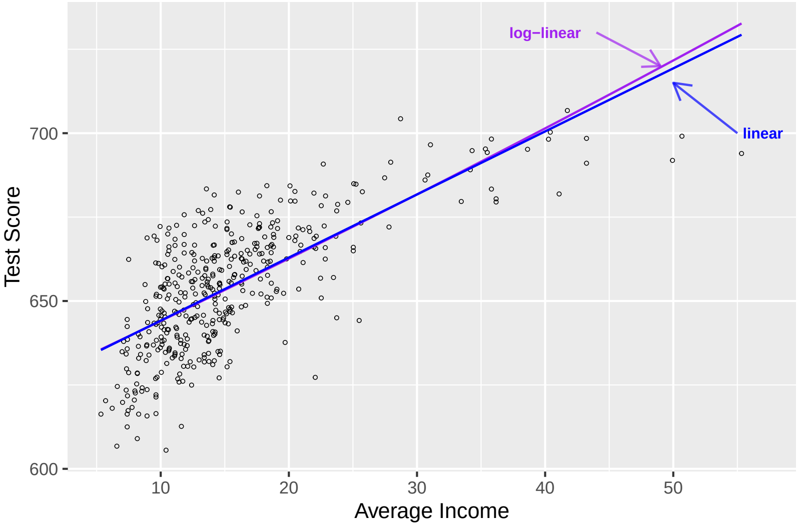
Example: the Log(TestScore) – Income relation

- The OLS regression of ln(Income) on Testscore yields

$$ln(\widehat{TestScore}) = \underset{(0.0028)}{6.439} + \underset{(0.0002)}{0.003} \times (Income)$$

- **Interpretation** of β_1: a unit increase in `Income` (here is $1000) is associated with an increase in TestScore of 0.3% in `TestScore`.
- **Calculate on the mean values**: Then if the mean of `TestScore` is 650, the predicted change in test score from a $1000 increase in income is $0.3\% \times 650 = 1.95$ points.

Test scores: log-linear function



The most common use of log-linear functions

- **Mincer Earning Function**:

$$ln(Earnings) = \beta_0 + \beta_1 SchoolingYears + \beta_2 Experience + \beta_3 Experience^2 + u$$

- It is widely used in labor economics to study the relationship between **earnings** and **human capital**.
  - β_1 is called as **the rate of the returns to schooling**.
- Suppose our estimated equation of Mincer Earning Function is:

$$ln(Earnings) = 2.881 + 0.06 SchoolingYears + 0.01 Experience - 0.0003 Experience^2 + u$$

- **Question**: *How to interpret the meaning of* $\hat{\beta}_1 = 0.06$*?*
- **Answer**: *A one-year increase in schooling is associated with a* 0.06% *increase in earnings.*

Case III. Log-log population regression function

- the regression model is

$$ln(Y_i) = \beta_0 + \beta_1 ln(X_i) + u_i$$

- Change X:

$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)]$$

Case III. Log-log population regression function

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$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)]$$
$$\Rightarrow ln(1 + \frac{\Delta Y}{Y}) = \beta_1 ln(1 + \frac{\Delta X}{X})$$

Case III. Log-log population regression function

- **the regression model is**

$$ln(Y_i) = \beta_0 + \beta_1 ln(X_i) + u_i$$

- **Change X:**

$$ln(\Delta Y + Y) - ln(Y) = [\beta_0 + \beta_1 ln(X + \Delta X)] - [\beta_0 + \beta_1 ln(X)]$$
$$\Rightarrow ln(1 + \frac{\Delta Y}{Y}) = \beta_1 ln(1 + \frac{\Delta X}{X})$$
$$\Rightarrow \frac{\Delta Y}{Y} \cong \beta_1 \frac{\Delta X}{X}$$

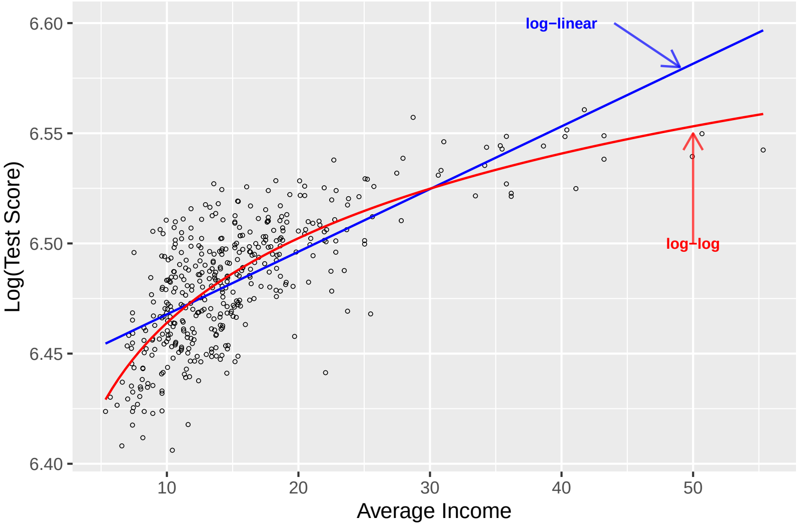
- **Now** $100\frac{\Delta Y}{Y} = percentage\ change\ in\ Y$ **and** $100\frac{\Delta X}{X} = percentage\ change\ in\ X$
- **Therefore a** 1% **change in X by one unit is associated with a** β_1**% change in Y,thus** β_1 **has the interpretation of an elasticity**.

Test scores and income: log-log specifications

$$\widehat{ln(TestScore)} = 6.336 + 0.055 \times ln(Income)$$
$$(0.006) \quad (0.002)$$

- **Interpretation of β_1: A 1% increase in Income is associated with an increase of 0.055% in TestScore.**
- **Calculate on the mean values: Then if the means of `TestScore` and `Income` are 650 and $\$/25,000$, the predicted change in `TestScore` from a $\$250(1\%)$ increase in `Income` is 3.575 points($650 \times 0.055\% = \$3.575$).**

Test scores: The log-linear and log-log functions



Test scores: The linear-log and cubic functions

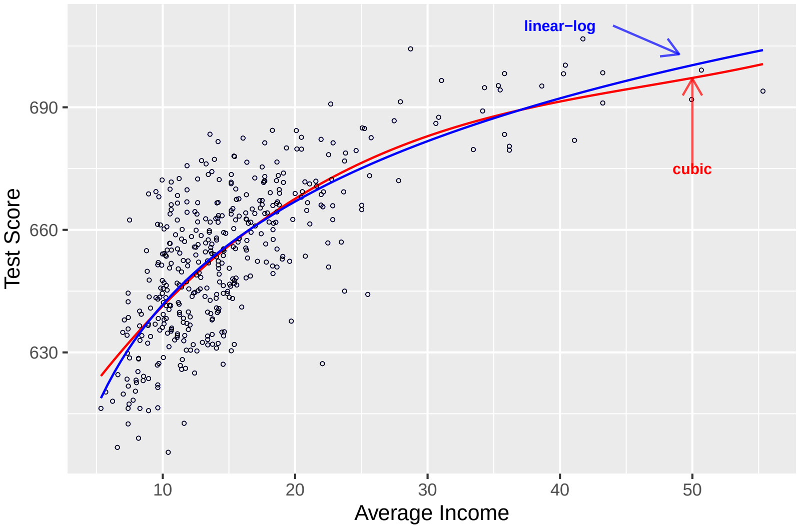


Table 3: Test Score and Income

	Dependent Variable: Test Score		
	testscr	log.testscr	testscr
	(1)	(2)	(3)
loginc	36.420***	0.055***	
		(0.002)	
avginc			5.019***
			(0.704)
I(avginc^2)			−0.096***
			(0.029)
I(avginc^3)			0.001**
			(0.0003)
Constant	557.832***	6.336***	600.079***
	(0.003)	(0.006)	(5.078)
Observations	420	420	420
Adjusted R^2	0.561	0.557	0.555
Residual Std. Error	12.618	0.019	12.707
F Statistic	537.444***	527.238***	175.352***

Choice of specification: guided by economic theory

- When regression functions yield similar results, how do you choose between them?
- Consider these guidelines:
    1. Apply economic theory (Choose the model with the most sensible interpretation for your context)
    2. Rely on **t-tests** and **F-tests** for statistical validation (More complex tests are rarely necessary)
    3. Evaluate model fit through visual inspection of predicted values and metrics like $\overline{R^2}$ or SER

Wrap Up

- Adding polynomial terms to significant variables allows for single and joint significance testing. Incorporate **quadratic or cubic terms** when they prove statistically significant.
- **Logarithmic transformations** effectively capture nonlinear relationships between variables.
- While pinpointing the exact cause of functional form misspecification is often difficult in practice.
- In economic analysis, **logarithms** and **polynomial terms** (quadratic or cubic) are generally **sufficient** to model most important nonlinear relationships.

Interactions Between Independent Variables

Introduction

- Try to answer following question:
  - *how the effect on Y of a change in an independent variable X depends on the value of another independent variable Z.*
- The question will be answered by putting an *interaction*, which is *the product of two independent variables*, in a regression.
  - The term is called an interaction term.
- Consider three cases:
  1. Interactions between **two binary variables**.
  2. Interactions between **a binary and a continuous variable**.
  3. Interactions between **two continuous variables**.

Interactions Between Two Binary Variables

- Assume we would like to study the earnings of worker in the labor market
- The population linear regression of Y_i is

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + u_i$$

- Dependent Variable: **log earnings**(Y_i,where $Y_i = ln(Earnings)$)
- Independent Variables: two binary variables
  - $D_{1i} = 1$ if the person graduate from college,$D_{1i} = 0$ otherwise
  - $D_{2i} = 1$ if the worker's gender is female,$D_{2i} = 0$ otherwise

- Question: how to interpret the coefficients β_1 and β_2 of the model?
  - β_1 is the effect of having a college degree on log earnings, *holding gender constant*
  - β_2 is the effect of being female on log earnings, *holding schooling constant*.

Interactions Between Two Binary Variables

- In this specification, the effect of a college degree on earnings is **identical** for men and women when gender is held constant, thus β_1 for both men and women.
- However, this assumption may not be realistic.
  - A college degree could have different impact on earnings for men versus women. In other words, the effect of a college degree on earnings is not the same for men and women.
- To capture the difference in the **effect of a college degree on earnings** for men and women, we need to modify our regression model.

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 (D_{1i} \times D_{2i}) + u_i$$

- The product term $D_{1i} \times D_{2i}$ is called an **interaction term** or **interacted regressor**.

Interactions Between Two Binary Variables:

- The population regression model now is

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 (D_{1i} \times D_{2i}) + u_i$$

- Let d_2 be the value of D_{2i},which is 0(Men) or 1(Women).
- Then the expected earnings of men and women **without a college degree** are

$$E(Y_i|D_{1i} = 0, D_{2i} = d_2) = \beta_0 + \beta_1 \times 0 + \beta_2 d_2 + \beta_3(0 \times d_2) = \beta_0 + \beta_2 d_2$$

- Then the expected earnings of men and women **with a college degree** are

$$\begin{aligned} E(Y_i|D_{1i} = 1, D_{2i} = d_2) &= \beta_0 + \beta_1 \times 1 + \beta_2 d_2 + \beta_3(1 \times d_2) \\ &= \beta_0 + \beta_1 + \beta_2 d_2 + \beta_3 d_2 \end{aligned}$$

Interactions Between Two Binary Variables:

- The effect of a college degree on earnings is the difference of expected values,which is

$$E(Y_i|D_{1i}=1,D_{2i}=d_2)-E(Y_i|D_{1i}=0,D_{2i}=d_2)=\beta_1+\beta_3 d_2$$

- In the binary variable interaction specification, the effect of acquiring a college degree (a unit change in $D_{1i} from 0 to 1$) depends on the person's gender(D_{2i}).
  - If the person is male,thus $D_{2i}=d_2=0$,then the effect is β_1
  - If the person is female,thus $D_{2i}=d_2=1$,then the effect is $\beta_1+\beta_3$
- Therefore, the coefficient β_3 is **the difference** in *the effect of acquiring a college degree on earnings* for women versus men.

Application: the STR and the English learners

- **Let $HiSTR_i$ be a binary variable for STR**
  - $HiSTR_i = 1$ **if the** $STR \geq 20$
  - $HiSTR_i = 0$ **otherwise**
- **Let $HiEL_i$ be a binary variable for the share of English learners**
  - $HiEL_i = 1$ **if the** $el_{pct} \geq 10 percent$
  - $HiEL_i = 0$ **otherwise**

Application: the STR and the English learners

- the OLS regression result is

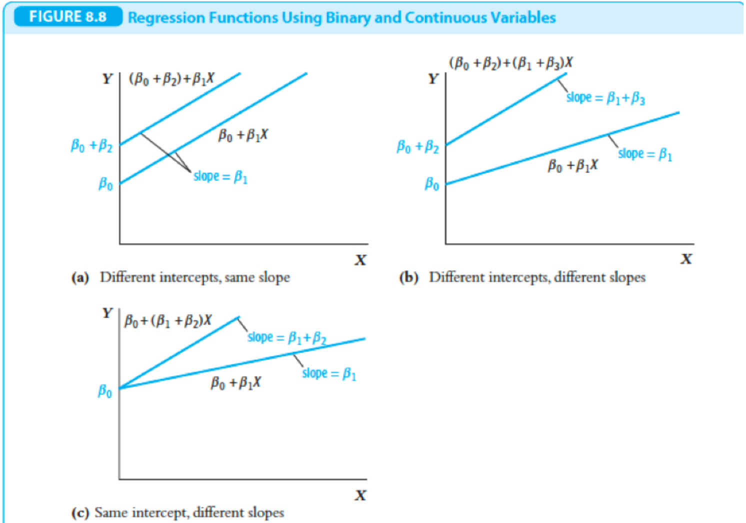
$$\widehat{TestScore} = \underset{(11.9)}{682.2} - \underset{(0.59)}{0.97}STR + \underset{(19.5)}{5.6}HiEL - \underset{(0.97)}{1.28}(STR \times HiEL)$$

- **Question**: *What does the interaction coefficient of β_3 here mean?*
- **Answer**: The effect of class size on test scores varies between the "higher-share-immigrant" class and the "lower-share immigrants" class.
- **Answer**: The performance gap in test scores between large class($STR > 20$) and small class($STR \leq 20$) varies between the "higher-share-immigrant" class and the "lower-share immigrants" class.
- More precisely,the gap of test scores is **positively** related with the "higher-share-immigrant" class though **insignificantly**.

Interactions: a Continuous and a Binary Variable

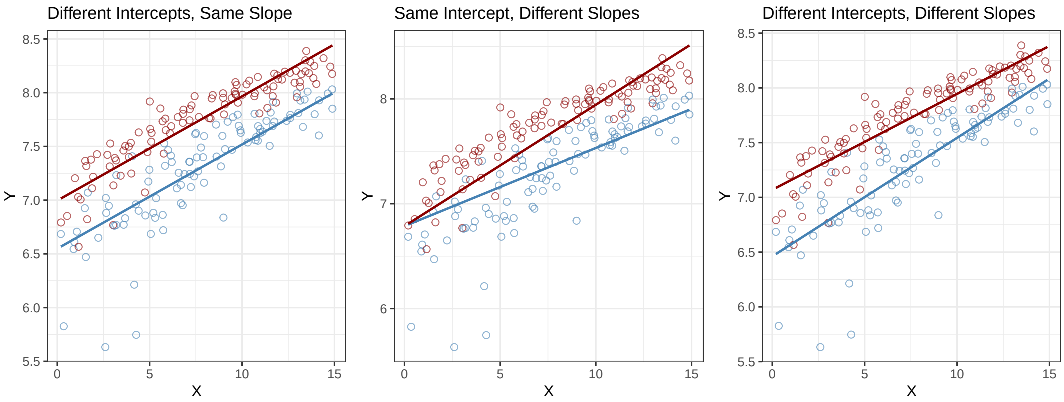
- **Binary Variable**: eg, whether the worker has a college degree (D_i)
- **Continuous Variable**: eg, the individual's years of work experience (X_i)
- In this case, we can have three specifications:
  1. No interaction
  $$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + u_i$$
  2. A interaction and only one independent variable
  $$Y_i = \beta_0 + \beta_1 X_i + \beta_2 (D_i \times X_i) + u_i$$
  3. A interaction and two independent variables
  $$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 (D_i \times X_i) + u_i$$

A Continuous and a Binary Variable: Three Cases



Interactions of binary variables and continuous variables can produce three different population regression functions: (a) $\beta_0 + \beta_1 X + \beta_2 D$ allows for different intercepts but has the same slope, (b) $\beta_0 + \beta_1 X + \beta_2 D + \beta_3(X \times D)$ allows

A Continuous and a Binary Variable: Three Cases



A Continuous and a Binary Variable: Specifications

- All three specifications are just different versions of the multiple regression model.
- Different specifications are based on different assumptions of the relationships of X on Y depending on D.
- The **Model 3** is preferred, because it allows for both different intercepts and different slops.

Application: the STR and the English learners

- $HiEL_i$ is still a binary variable for English learner, while $\texttt{STR}$ is a continuous variable for class size.
- The estimated interaction regression

$$\widehat{TestScore} = \underset{(11.9)}{682.2} - \underset{(0.59)}{0.97}STR + \underset{(19.5)}{5.6}HiEL - \underset{(0.97)}{1.28}(STR \times HiEL)$$

- For districts with a low fraction of English learners,thus $HiEL_i = 0$,the estimated regression line is

$$\widehat{TestScore} = 682.2 - 0.97STR_i$$

- For districts with a high fraction of English learners,thus $HiEL_i = 1$,the estimated regression line is

$$\widehat{TestScore} = 682.2 + 5.6 - 0.97STR_i - 1.28STR_i = 687.8 - 2.25STR_i$$

Application: the STR and the English learners

- The difference between these two effects, 1.28 points, is the coefficient on the interaction term.
- The value of β_3 here(-1.28) means that *the effect of class size on test scores varies between the "higher-share-immigrant" class and the "lower-share immigrants or more native" class.*
- More precisely,negatively related with the "higher-share-immigrant" class though insignificantly.

Testing Model Specifications

- We can evaluate the three specifications using **F-tests** and **t-tests**.

1. Testing whether test scores are identical between groups (same intercept and slope)
   - $H_0 : \beta_2 = \beta_3 = 0$
   - F-test result: F-statistic = 89.9, significant at the 1% level, allowing us to reject this hypothesis.
2. Testing whether effects are identical between groups (same slope but potentially different intercepts)
   - $H_0 : \beta_3 = 0$
   - t-statistic = -1.32, not significant at the 10% level.

Further Testing

3. Testing whether groups have the same intercept but different slopes
   - $H_0 : \beta_2 = 0$
   - t-statistic = 0.29, not significant at the 10% level.

- The high correlation between regressors $HiEL$ and $STR * HiEL$ leads to inflated standard errors for individual coefficients.
- **Conclusion**: While we cannot determine which specific coefficient is non-zero, we have strong evidence to reject the hypothesis that both coefficients equal zero.

Interactions Between Two Continuous Variables

- Now suppose that both independent variables (X_{1i} and X_{2i}) are continuous.
  - X_{1i} is his or her years of work experience
  - X_{2i} is the number of years he or she went to school.
- there might be an interaction between these two variables so that the effect on wages of an additional year of experience depends on the number of years of education.
- the population regression model

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 (X_{1i} \times X_{2i}) + u_i$$

Interactions Between Two Continuous Variables

- Thus the effect on Y of a change in X_1, holding X_2 constant, is

$$\frac{\Delta Y}{\Delta X_1} = \beta_1 + \beta_3 X_2$$

- A similar calculation shows that the effect on Y of a change ΔX_1 in X_2, holding X_1 constant, is

$$\frac{\Delta Y}{\Delta X_2} = \beta_2 + \beta_3 X_1$$

- That is, if X_1 changes by ΔX_1 and X_2 changes by ΔX_2, then the expected change in Y

$$\Delta Y = (\beta_1 + \beta_3 X_2)\Delta X_1 + (\beta_2 + \beta_3 X_1)\Delta X_2 + \beta_3 \Delta X_1 \Delta X_2$$

Application: the STR and the English learners

- **Now** `STR` **and** `PctEL` **are both continuous variables.**
- **The estimated interaction regression**

$$\widehat{ln(TestScore)} = \underset{(11.8)}{686.3} - \underset{(0.059)}{1.12}STR - \underset{(0.037)}{0.67}PctEL + \underset{(0.019)}{0.0012}(STR \times PctEL)$$

- **Interpretation of** β_3: *how the effect of class size on test scores varies along with the share of English learners in the class.*
- More precisely, *increase 1 unit of the share of English learners* make **the effect of class size on test scores** increase extra 0.0012 scores.

Application: the STR and the English learners

- when the percentage of English learners is at the **median**($PctEL = 8.85$), the slope of the line relating test scores and the STR is

$$\frac{\Delta Y}{\Delta X_1} = \beta_1 + \beta_3 X_2 = -1.12 + 0.0012 \times 8.85 = -1.11$$

- when the percentage of English learners is at the **75th percentile**($PctEL = 23.0$), the slope of the line relating test scores and the STR is

$$\frac{\Delta Y}{\Delta X_1} = \beta_1 + \beta_3 X_2 = -1.12 + 0.0012 \times 23.0 = -1.09$$

- The effect of class size on test scores depends on the share of English learners.
- However, the difference between these estimated effects is not statistically significant.
  - The t-statistic testing whether the coefficient on the interaction term is zero is 0.06, which is **not statistically significant** at the 10% level.

Application: STR and Test Scores in a Summary

- Although these nonlinear specifications extend our knowledge about the relationship between STR and Testscore, it must be augmented with control variables such as **economic background** to avoid OVB bias.
- Two measures of the economic background of the students:
    1. the percentage of students eligible for a subsidized lunch
    2. the logarithm of average district income.

Application: STR and Test Scores in a Summary

- Then three specific questions about test scores and the student–teacher ratio.
  1. After controlling for differences in economic characteristics, does the effect on test scores of STR depend on the fraction of English learners?
  2. Does this effect depend on the value of the student–teacher ratio(STR)?
  3. Most important, after taking economic factors and nonlinearities into account,what is the estimated effect on test scores of reducing the student–teacher ratio by **1** students per teacher?

	score						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
str	-1.00^{***}	-0.73^{**}	-0.97	-0.53	64.34^{**}	83.70^{**}	65.29^{**}
	(0.27)	(0.26)	(0.59)	(0.34)	(24.86)	(28.50)	(25.26)
I(str^2)					-3.42^{**}	-4.38^{**}	-3.47^{**}
					(1.25)	(1.44)	(1.27)
I(str^3)					0.06^{**}	0.07^{**}	0.06^{**}
					(0.02)	(0.02)	(0.02)
str:HiEL			-1.28	-0.58		-123.28^{*}	
			(0.97)	(0.50)		(50.21)	
I(str^2):HiEL						6.12^{*}	
						(2.54)	
I(str^3):HiEL						-0.10^{*}	
						(0.04)	
english	-0.12^{***}	-0.18^{***}					-0.17^{***}
	(0.03)	(0.03)					(0.03)
HiEL			5.64	5.50	-5.47^{***}	816.08^{*}	
			(19.51)	(9.80)	(1.03)	(327.67)	
lunch	-0.55^{***}	-0.40^{***}		-0.41^{***}	-0.42^{***}	-0.42^{***}	-0.40^{***}
	(0.02)	(0.03)		(0.03)	(0.03)	(0.03)	(0.03)
log(income)		11.57^{***}		12.12^{***}	11.75^{***}	11.80^{***}	11.51^{***}
		(1.82)		(1.80)	(1.77)	(1.78)	(1.81)
Constant	700.15^{***}	658.55^{***}	682.25^{***}	653.67^{***}	252.05	122.35	244.81
	(5.57)	(8.64)	(11.87)	(9.87)	(163.63)	(185.52)	(165.72)
N	420	420	420	420	420	420	420
Adjusted R^2	0.77	0.79	0.31	0.79	0.80	0.80	0.80

$^{*}p < .05$; $^{**}p < .01$; $^{***}p < .001$
Robust S.E. are shown in the parentheses

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str:HiEL			-1.28	-0.58		-123.28^{*}	
			(0.97)	(0.50)		(50.21)	
I(str^2):HiEL						6.12^{*}	
						(2.54)	
I(str^3):HiEL						-0.10^{*}	
						(0.04)	
english	-0.12^{***}	-0.18^{***}					-0.17^{***}
	(0.03)	(0.03)					(0.03)
HiEL			5.64	5.50	-5.47^{***}	816.08^{*}	
			(19.51)	(9.80)	(1.03)	(327.67)	
lunch	-0.55^{***}	-0.40^{***}		-0.41^{***}	-0.42^{***}	-0.42^{***}	-0.40^{***}
	(0.02)	(0.03)		(0.03)	(0.03)	(0.03)	(0.03)
log(income)		11.57^{***}		12.12^{***}	11.75^{***}	11.80^{***}	11.51^{***}
		(1.82)		(1.80)	(1.77)	(1.78)	(1.81)
Constant	700.15^{***}	658.55^{***}	682.25^{***}	653.67^{***}	252.05	122.35	244.81
	(5.57)	(8.64)	(11.87)	(9.87)	(163.63)	(185.52)	(165.72)
N	420	420	420	420	420	420	420
Adjusted R^2	0.77	0.79	0.31	0.79	0.80	0.80	0.80

$^{*}p < .05$; $^{**}p < .01$; $^{***}p < .001$
Robust S.E. are shown in the parentheses

	score						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
str	-1.00^{***}	-0.73^{**}	-0.97	-0.53	64.34^{**}	83.70^{**}	65.29^{**}
	(0.27)	(0.26)	(0.59)	(0.34)	(24.86)	(28.50)	(25.26)
I(str^2)					-3.42^{**}	-4.38^{**}	-3.47^{**}
					(1.25)	(1.44)	(1.27)
I(str^3)					0.06^{**}	0.07^{**}	0.06^{**}
					(0.02)	(0.02)	(0.02)
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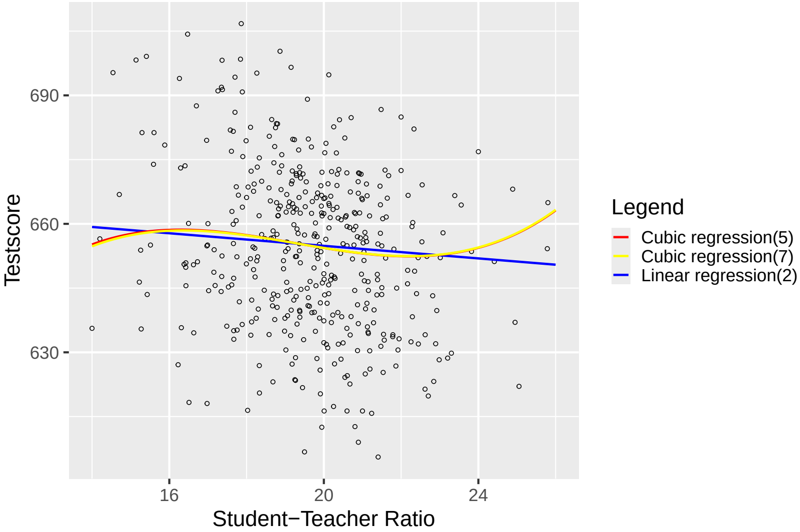
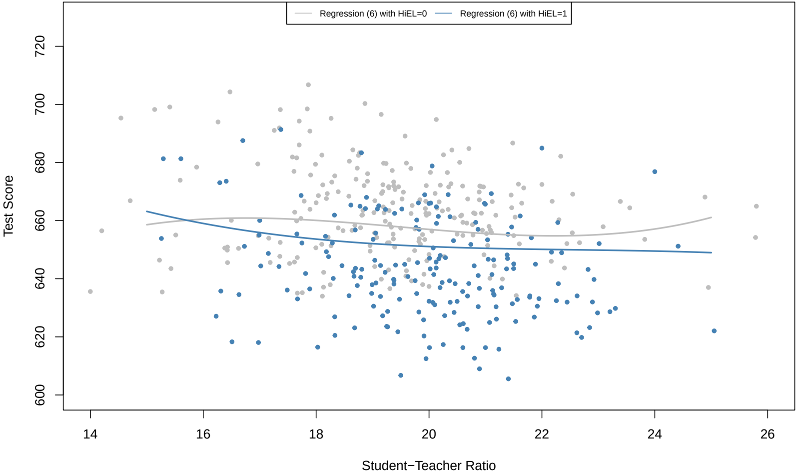
Three Regressions on graph



Interaction on graph



Warp up

- Interaction term is a powerful tool to study the effect of one variable on another, while controlling for other variables.
- In one case, **interaction term with high order polynomial terms** are used to model nonlinear relationship to avoid **misspecification**.
  - In this case, the coefficient of interaction term is not of interest.(interaction terms as control variables)
- In another case, **interaction term** is used to study the effect of one variable on another, while controlling for other variables.
  - In this case, the coefficient of interaction term is of interest as **Difference-in-Difference** estimator.

A Lastest and Smart Application: Jia and Ku(2019)

Jia and Ku(2019)

- Ruixue Jia and Hyejin Ku, "Is China's Pollution the Culprit for the Choking of South Korea?Evidence from the Asian Dust",The Economic Journal,Volume 129, Issue 624, November 2019, Pages 3154–3188.
- **Main Question**:
  - Does air pollution from China spill over to South Korea, and if so, how does it affect the health of South Koreans?

Empirical Strategy

- A naive strategy:
  - Dependent variable: **Deaths in South Korea** (*respiratory and cardiovascular mortality*)
  - Independent variable: **Chinese pollution**(*Air Quality Index*)

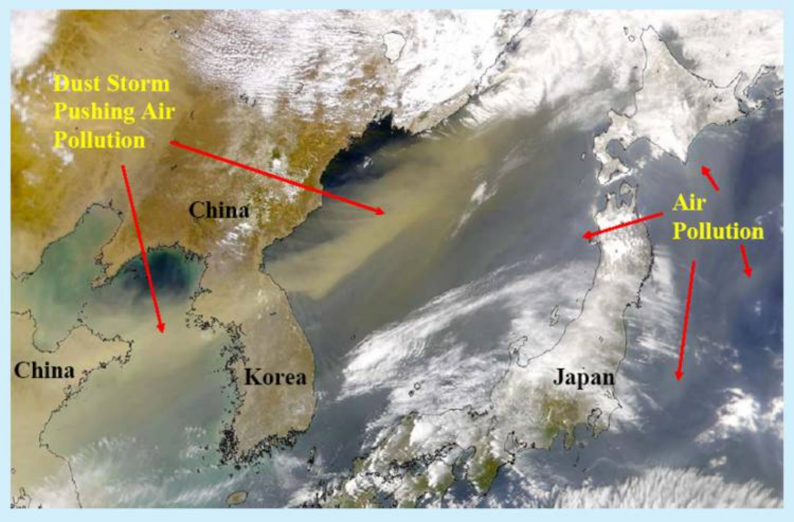
$$Mortality_{ijk} = \beta_0 + \beta_1 ChinesePollution_{jk} + \delta X_{ijk} + u_{ijk}$$

- Because the observed or measured air pollution in Seoul increases in periods when China is more polluted does not mean that the pollution must have **originated from China**.
- u_{ijk} can be correlated with $ChinesePollution_{jk}$, like Korean's local activities, leading to **endogeneity**.
- **Question**: *Can we find a exogenous variable that is uncorrelated with* $ChinesePollution_{jk}$ *but correlated with* $Mortality_{ijk}$*?*
- **Answer**: Yes, it is **Asian Dust**.

Jia and Ku(2019): Asian Dust as a carrier of pollutants

- **Asian Dust** (also yellow dust, yellow sand, yellow wind or China dust storms) is a **meteorological phenomenon** which affects much of East Asia year round but especially during the **spring months**.
  - The dust originates in the deserts of Mongolia, Kazakhstan and China(Inner Mongolia),where high-speed surface winds and intense dust storms kick up dense clouds of fine, dry soil particles.
  - These clouds are then carried **eastward** by prevailing winds and pass over China, North and South Korea, and Japan, as well as parts of the Russian Far East.
  - In recent decades,Asian Dust brings with **pollution as well as its by-products** most originated from China.

Jia and Ku(2019): Asian Dust as a carrier of pollutants



Jia and Ku(2019): Asian Dust as a carrier of pollutants

- Key features of Asian Dust for this paper:
  1. A clear directional aspect where winds transport Chinese pollutants to Korea but **not vice versa**.
  2. **Exogenous** to South Korea's local economic activities, with wind patterns and topography creating rich spatial and temporal **variation** in its occurrence.
  3. Due to its visual prominence, Asian dust is **monitored and recorded** at monitoring stations throughout South Korea.

Jia and Ku(2019):Estimation Strategy

- **Interaction Variable**: **Asian dust**(*the number of Asian dust days in South Korea*)
- **Control Variables**: Time, Regions, Weather,Local Economic Conditions
- The impact of **Chinese pollution** on district-level *mortality* that operates via **Asian dust**

$$
\begin{aligned}
Mortality_{ijk} = \beta_0 + \beta_1 AsianDust_{ijk} + \beta_2 ChinesePollution_{jk} \\
+ \beta_3 AsianDust_{ijk} \times ChinesePollution_{jk} + \delta_1 X_{ijk} + u_{ijk}
\end{aligned}
$$

- Main coefficient of interest is β_3, which measures the effect of Chinese pollution in year j and month k **carried by Asian Dust** on mortality in district i of South Korea.

Jia and Ku(2019): the result of interaction terms

Table 2: The Impact of Dust*China's Pollution on Mortality Rates in South Korea

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
				Baseline				Placebo Tests	
			Mortality rates: Respiratory and Cardiovascular					Cancers	Accidents
Mean Dependent Var.				12.23				16.30	4.21
#Dust*China's Mean AQI				0.038**	0.033*	0.043**	0.040**	-0.008	0.007
				(0.016)	(0.018)	(0.018)	(0.019)	(0.020)	(0.009)
#Dust	0.076***		0.039	-0.251*	-0.214	-0.313**	0.072	0.525*	-0.102
	(0.025)		(0.032)	(0.131)	(0.142)	(0.149)	(0.240)	(0.275)	(0.119)
China's Mean AQI		0.265***	0.193*	0.117	0.138	0.202*	0.200*	-0.080	0.005
		(0.081)	(0.104)	(0.105)	(0.107)	(0.110)	(0.111)	(0.121)	(0.060)
District FE*Year FE	Y	Y	Y	Y	Y	Y	Y	Y	Y
Province FE*Month FE	Y	Y	Y	Y	Y	Y	Y	Y	Y
Weather (cubic polynomial)					Y	Y	Y	Y	Y
Local Prod (export, energy)						Y	Y	Y	Y
Local Prod*Dust							Y	Y	Y
Observations	29,464	29,464	29,464	29,464	28,952	28,024	28,024	28,024	28,024
R-squared	0.695	0.695	0.695	0.696	0.703	0.717	0.717	0.718	0.473

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Province FE*Month FE	Y	Y	Y	Y	Y	Y	Y	Y	Y
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Province FE*Month FE	Y	Y	Y	Y	Y	Y	Y	Y	Y
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Jia and Ku(2019): Placebo Test

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Summary

Wrap up

- We extend our multiple ols model form linear to nonlinear in Xs(the independent variables)
  - Polynomials,Logarithms and Interactions
  - The multiple regression framework can be extended to handle regression functions that are nonlinear in one or more Xs.
  - the difference between a standard multiple OLS regression and a nonlinear OLS regression model in Xs is mainly how to explain estimating coefficients.
- All of these are very useful and common tools with OLS regressions. It's important to understand them clearly.